Beauty Contests and Fat Tails in Financial Markets

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Abstract

This study seeks to explain the emergence of fat-tailed distributions of trading volumes and asset returns in financial markets. We use a rational expectations form of the herding model. In the model, traders infer other traders' private signals regarding the value of an asset by observing their aggregate buying actions. The rational expectations equilibrium outcome entails an upward sloping demand curve. This is because the information contained in others' signals is more encouraging than is reflected in the incremental price. That is, there are strategic complementarities in informed traders' buying actions. In this environment, we show that equilibrium trading volumes and asset returns follow

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fat-tailed distributions without making any parametric assumptions on private signals. Specifically, we demonstrate that the trading volume follows a power-law distribution when the number of traders is large and the signal is noisy. Furthermore, we provide simulation results to show that our model successfully reproduces the observed distributions of daily stock returns.
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1 Introduction

A traditional economic explanation for the excess volatility of trading volumes and returns of financial markets relies on rational herd behavior by traders. In a situation where traders' action space is coarser than their private state space, their observable actions only partially reveal privately held information regarding the value of an asset. This property makes it possible for a single trader's action to cause an avalanche of similar actions by other traders. The idea of a chain reaction through the revelation of private information has been extensively discussed in the literature on herd behavior, informational cascades, and information aggregation. However, there have been few attempts to use herd behavior to explain fat-tailed distributions of stock return fluctuations.

That stock returns exhibit fat-tailed and leptokurtic distributions has been well established since Mandelbrot [22] and Fama [12]. The distribution of high-frequency stock returns clearly deviates from the normal distribution in its tail, and often decays slower than an exponential distribution. For example, Jansen and de Vries [17] have shown empirically that stock returns have a distribution with its tail decaying as a power function with the order of 3 to 5, which indicates that the fourth moment of the returns deviates substantially from a normal distribution. Such a fat tail and a high kurtosis have been regarded as a cue for understanding the excess volatility of stock returns.

This study shows that herd behavior generates fat tails in asset return distributions. To this end, we propose a model consisting of informed and uninformed traders and an auctioneer. The model closely builds on Minehart and Scotchmer [24] and Bru and Vives [7]. There are a large number of informed traders who receive imperfect private signals on the true value of an asset. The informed traders simultaneously choose between buying one unit of the asset or not buying the asset at all. To simplify the model, we depart from Glosten and Milgrom [14] or Smith [28], by assuming that informed traders cannot short-sell. We define a rational expectations equilibrium in which each trader submits their demand schedule conditional on the price of the asset. The rational choice made by an informed trader is based on the private signal they receive as well as the information revealed by other traders' actions through the equilibrium price. The price is set by an auctioneer, who aggregates the demand of the informed traders and matches it with the supply schedule submitted by uninformed traders. We show that, in this setting, the more that informed traders choose to buy the asset, the higher the asset price, which in turn signals higher asset value. As a result, a single trader's buying action induces buying actions by other traders who would not have bought otherwise. In this way, traders' strategies exhibit complementarity, and their actions are positively correlated.

The main theoretical contribution of our study is that the probability distribution of the equilibrium number of buying traders is shown to exhibit a power-law tail with an exponent of 0.5. A random variable M is said to follow a power-law distribution with exponent α if $\Pr(M > m)$ is proportional to $m^{-\alpha}$ for large values of m. In general, a power law with exponent α implies that any k-th moment of M is infinite for $k \geq \alpha$. Thus, with exponent 0.5, the equilibrium number of buying traders does not have a finite variance or mean. This indicates that the stochastic herd size in our equilibrium exhibits large volatility. The volatile herding of traders results in equilibrium asset price volatility.

Our model is similar to Keynes's beauty contest in terms of the way it describes herding behavior. Each trader recognizes that other traders possess private information equally valuable to their own. Therefore, each trader seeks to mimic the average trader. However, this behavior leads to a fragile equilibrium. Unlike the models that lead to indeterminate equilibria, the equilibrium in our model is locally unique, because we assume that traders' actions are discrete. This allows us to quantitatively characterize fluctuations in trading volumes and prices. The fluctuations are caused by randomness in private signals. Our analysis demonstrates that a power-law distribution of trading volumes emerges naturally in this setup.

Our study is related to the theoretical and empirical literature on imitative behavior in financial markets. Scharfstein and Stein [27], Banerjee [4], and Bikhchandani, Hirshleifer, and Welch [5] have developed models of herd behavior and informational cascades. These models have been employed in a number of studies to examine financial market crashes, including those by Caplin and Leahy [9], Lee [20] and Chari and Kehoe [10]. However, herding behavior in these studies is all-or-nothing herding because of a particular type of information structure they assume—sequential trading. As a result, few studies in this literature address the issue of stochastic financial fluctuations. An exception is the study by Gul and Lundholm [15], who demonstrated the emergence of stochastic herding by endogenizing traders' choice of waiting time. We follow this approach and focus on the stochastic aspect of financial fluctuations, but deviate from it by employing a model in which traders move simultaneously, and the equilibrium number of traders exhibits stochastic fluctuations. Our model of stochastic herding contributes to the literature by showing that informational cascades can generate not only extremely large fluctuations in trading volumes and prices but also an empirically relevant regularity regarding the frequency distribution of these fluctuations, which is summarized by power-law distributions.

While many statistical models are capable of replicating the power-law distribution of asset returns, few economic models have been developed for the same purpose. An important exception is the model developed by Gabaix et al. [13]. They offer empirical evidence of the power laws for trading volumes and asset returns, and provide a model which accounts for those distributions by making use of Zipf's law for firm sizes, which is a different power law from another context. Specifically, they argue that if the amount of funds managed by traders follows a power law, trading volumes and price changes also follow power laws. In contrast to Gabaix et al. [13], the present study does not rely on heterogeneity across traders in accounting for power laws in financial fluctuations. Instead, we assume that traders are homogeneous in size and in other respects. We show that, even in this symmetric setting, the interaction of a large number of traders generates stochastic herding to varying degrees (i.e., the number of traders who decide to buy the asset differs), thereby generating power laws in financial fluctuations. In so doing, we provide a new explanation for power laws in financial fluctuations which is complementary to the one advocated by Gabaix et al. [13].[1]

[1] Another area to which this study, especially the technical section, is related is the literature on critical phenomena in statistical physics. A number of statistical physicists have investigated the empirical fluctuations of financial markets (surveys of these studies can be found in Bouchaud and Potters [6] and Mantegna and Stanley [23]), and some studies in this literature reproduce the observed power laws by applying a methodology often used for the analysis of critical phenomena to herd

The remainder of the study is organized as follows. Section 2 presents the model. Section 3 analytically shows that a power-law distribution emerges for trading volumes when the number of traders tends to infinity, and provides an intuition for the mechanism behind it. Section 4 presents numerical simulations to show that the equilibrium volumes follow a power law under a finite number of traders, and that the equilibrium return distribution matches its empirical counterpart. Section 5 concludes.

2 Model

2.1 Model and equilibrium

There are two states of the world, $s = H, L$. Throughout the study, we will assume that the true state is H, unless stated otherwise. There is a common prior belief $\Pr(H) = \Pr(L) = 1/2$. Informed trader i receives an imperfect and private signal $X_{\delta,i}$ of the state. The signal is private in the sense that each trader does not observe other traders' signals. Also, the signal is imperfect in the sense that $X_{\delta,i}$ does not fully reveal the true state. Signal $X_{\delta,i}$ is identically and independently distributed across i with conditional cumulative distribution function F_δ^s for $s = H, L$ with common bounded support Σ. Let $\bar{\Sigma}$ denote the upper bound of Σ. We assume that F_δ^s has a continuous, strictly positive-valued density f_δ^s over Σ for $s = H, L$. We order the signal

behavior models (Bak, Paczuski, and Shubik [3]; Cont and Bouchaud [11]; Stauffer and Sornette [31]). However, these studies do not model traders' purposeful behavior and rational learning, and therefore fail to link their analyses to the existing body of financial economics literature. More importantly, these studies do not address why market activities exhibit criticality. This issue is important because, according to these studies, power laws in financial fluctuations typically occur only when the parameter that governs the connectivity of the networked traders takes a critical value. These two issues will be addressed in this study.

based on the monotone likelihood ratio property (MLRP) such that the likelihood ratio $\ell_\delta \equiv f_\delta^L / f_\delta^H$ is decreasing. MLRP holds for a signal with two states without loss of generality (Smith and Sørensen [29]). We assume that the likelihood ratio $\ell_\delta(x)$ satisfies $\sup_{x\in\Sigma} |\ell_\delta(x) - 1| < \delta$. This assumption implies that the informativeness of the signal is ordered by $\delta > 0$.

We further define likelihood ratios $\lambda_\delta(x) \equiv F_\delta^L(x)/F_\delta^H(x)$ and $\Lambda_\delta(x) \equiv (1-F_\delta^L(x))/(1-F_\delta^H(x))$. As shown by Smith and Sørensen [29], MLRP implies that, for any $x \in \Sigma$,

$$\lambda_\delta(x) > \ell_\delta(x) > \Lambda_\delta(x) > 0, \tag{1}$$

and that $\lambda_\delta(x)$ and $\Lambda_\delta(x)$ are strictly decreasing in x:

$$\frac{d\lambda_\delta(x)}{dx} = \frac{f_\delta^L(x)}{F_\delta^H(x)} - \frac{F_\delta^L(x) f_\delta^H(x)}{(F_\delta^H(x))^2} = \frac{f_\delta^H(x)}{F_\delta^H(x)} (\ell_\delta(x) - \lambda_\delta(x)) < 0, \tag{2}$$

$$\frac{d\Lambda_\delta(x)}{dx} = -\frac{f_\delta^L(x)}{1-F_\delta^H(x)} + \frac{(1-F_\delta^L(x)) f_\delta^H(x)}{(1-F_\delta^H(x))^2} = \frac{f_\delta^H(x)}{1-F_\delta^H(x)} (\Lambda_\delta(x) - \ell_\delta(x)) < 0 \tag{3}$$

We consider a series of markets, indexed by the number of informed traders $n = n_o, n_o + 1, \ldots$ in the market, where n_o is a large integer. Following Minehart and Scotchmer [24], we consider informed traders who simultaneously choose whether to buy one unit of an asset or to not buy the asset at all. The asset has a common value, and is worth 1 in state H and 0 in state L to all traders. The trading unit of the asset is given by $1/n$. Each informed trader submits their demand function $d_{n,i} : \mathbb{R}_+ \mapsto \{0, 1\}$ to an auctioneer. The demand function $d_{n,i} = d_{n,i}(p_n)$ describes whether trader i with private signal $x_{\delta,i}$ buys or not for each possible price of the asset, p_n, where $d_{n,i} = 1$ indicates buying and $d_{n,i} = 0$ not-buying. Aggregate demand expressed in terms of the trading unit is $D(p_n) = \sum_{i=1}^n d_{n,i}(p_n)/n$ that maps $\mathbb{R}_+$ to $\{0, 1/n, 2/n, \ldots, 1\}$.

Uninformed traders decide on whether to supply the asset depending only on p_n.[2] Let $S(p_n)$ denote the aggregate supply function of the uninformed traders. We assume

[2]The informational asymmetry between informed and uninformed traders in this model is similar

that the supply function is continuous, differentiable, and upward sloping ($S' > 0$). We assume that $S(0.5) = 0$. That is, the aggregate supply is 0 at the price level that reflects the common prior belief. Furthermore, we assume that $\bar{p} \equiv S^{-1}(1) < 1$. That is, even in an equilibrium in which all n informed traders decide to buy, the equilibrium price $\bar{p}$ does not achieve the maximum value of the asset, 1. Under this setup, the equilibrium price P_n^* has a bounded support $[0.5, \bar{p}]$.

Transactions are implemented by an auctioneer, who receives the demand functions $(d_{n,i}(\cdot))_{i=1}^n$ from the informed traders and the supply function $S(\cdot)$ from the uninformed traders, and chooses equilibrium price p_n^* such that $D(p_n^*) = S(p_n^*)$.[3] Let m_n^* denote the equilibrium number of buying traders, i.e., $m_n^* \equiv D(p_n^*)n$.

Each informed trader i computes by Bayes' rule his posterior belief $r_{n,i}$ that the state is H. Trader i forms the posterior belief using private signal $x_{\delta,i}$ and price p_n. Thus, the posterior belief is a mapping $r : \mathbb{R}_+ \times \Sigma \mapsto [0, 1]$ such that $r_{n,i} = r(p_n, x_{\delta,i})$. Informed traders are assumed to be risk-neutral and to maximize their subjective expected payoff. The expected payoff of a trader is 0 when $d_{n,i} = 0$ regardless of the belief, whereas it is $r_{n,i} - p_n$ when $d_{n,i} = 1$. Thus, trader i buys the asset if and only if $r_{n,i} \geq p_n$.

For each realization of a profile of private signals $(x_{\delta,i})_{i=1}^n$, a rational expectations equilibrium consists of the number of buying informed traders m_n^*, price p_n^*, demand functions $(d_{n,i})_{i=1}^n$, and the posterior belief r, such that (i) for any p_n, $d_{n,i}(p_n)$ maximizes

to event uncertainty, which was introduced by Avery and Zemsky [2] as a condition for herding to occur in financial markets.

[3]This mechanism of implementing a rational expectations equilibrium through the submission of demand schedules follows Bru and Vives [7]. Without information aggregation by the auctioneer, the model would become similar to that of Minehart and Scotchmer [24], who showed that traders cannot agree to disagree in a rational expectations equilibrium, i.e., an equilibrium may not exist, or if it exists, it is a herding equilibrium where all traders choose the same action.

trader i's expected payoff evaluated at $r_{n,i} = r(p_n, x_{\delta,i})$ for any i, (ii) $r_{n,i}$ is consistent with p_n and $x_{\delta,i}$ for any i, and (iii) the auctioneer delivers the orders $d^*_{n,i} = d_{n,i}(p^*_n)$, and clears the market, i.e., $S(p^*_n) = m^*_n/n$, where $m^*_n = \sum_{i=1}^{n} d^*_{n,i}$. Random variables $(X_{\delta,i}, P^*_n, M^*_n)$ are denoted by upper case letters, while their realizations by lower case letters $(x_{\delta,i}, p^*_n, m^*_n)$.

2.2 Traders' optimal strategy

We derive the optimal demand schedule of trader i as a threshold rule. When the auctioneer chooses p_n that satisfies $S(p_n) = m/n$, it reveals that there are m buying informed traders. Let $p_n(m)$ denote such a level of price. The optimal threshold rule is given by

$$d_{n,i}(p_n(m)) = \begin{cases} 1 & \text{if } x_{\delta,i} \geq \bar{x}_n(m), \\ 0 & \text{otherwise,} \end{cases}$$

for $m = 1, 2, \ldots, n$, where $\bar{x}_n(m)$ denotes the threshold level for the private signal at which a buying trader is indifferent between buying and not buying, given $p_n(m)$.

We solve for the optimal threshold $\bar{x}_n$ as follows. Given $p_n(m)$ under the threshold rule, and using functions λ_δ and Λ_δ, the likelihood ratios revealed by inaction ($d_{n,i} = 0$) and by buying ($d_{n,i} = 1$), respectively, are written as $\lambda_\delta(\bar{x}_n(m))$ and $\Lambda_\delta(\bar{x}_n(m))$.

Consider a trader making a buying bid at price $p_n(1)$. If this bid is struck by the auctioneer, this implies that the other $n-1$ informed traders do not bid at $p_n(1)$. Thus, the threshold is determined by

$$\frac{1}{p_n(1)} - 1 = \lambda_\delta(\bar{x}_n(1))^{n-1} \ell_\delta(\bar{x}_n(1)).$$

Similarly, each buying trader knows that, if the bid is executed at $p_n(m)$, there are $m - 1$ traders buying at $p_n(m)$ and $n - m$ traders not buying at $p_n(m)$. Then, the threshold $\bar{x}_n(m)$ is obtained by solving

$$\frac{1}{p_n(m)} - 1 = \lambda_\delta(\bar{x}_n(m))^{n-m}\Lambda_\delta(\bar{x}_n(m))^{m-1}\ell_\delta(\bar{x}_n(m)). \tag{4}$$

Equation (4) is the key to the subsequent analysis. The right-hand side shows the posterior private belief of a trader who receives signal $x_{\delta,i} = \bar{x}_n(m)$ and is buying at $p_n(m)$. Thus, this equation determines the threshold level of signal $\bar{x}_n(m)$ for which a trader is indifferent between buying and not-buying given $p_n(m)$. Note that this equation implicitly determines $\bar{x}_n(m)$ not only for integers but also any real number m.

Given the threshold behavior shown above, we obtain aggregate demand $D(p_n(m))$ by counting the number of informed traders with $x_{\delta,i} \geq \bar{x}_n(m)$ and dividing it by n. As a convention for the case of $m = 0$, we exogenously set as $p_n(0) = 0.5$ and $D(p_n(0)) = D(p_n(1))$. If $D(p_n(1)) = 0$ as a result of realized private signals, $p_n(0)$ clears the market since $D(p_n(0)) = D(p_n(1)) = S(p_n(0)) = 0$. If $D(p_n(1)) > 0$, $p_n(0)$ cannot clear the market.

With this setup, we obtain the following lemma stating that the aggregate demand curve is upward sloping when n is sufficiently large. This property holds because the more informed traders are buying the more signals in favor of H are revealed, and hence, the more likely each informed trader is to buy.[4]

Lemma 1 *There exists an n_o such that for any $n > n_o$, the threshold level of signal $\bar{x}_n(m)$ is strictly decreasing in m and the aggregate demand $D(p_n(m))$ is non-decreasing in m.*

[4]A similar result was presented in Nirei [25]. However, this study differs from it in that it used a Nash equilibrium, while the present study uses a rational expectations equilibrium. With the Nash formulation, the previous study was not able to establish the existence of equilibrium with a finite number of traders, which is accomplished in this study, as shown in Proposition 1.

Proof: By taking the total derivative of the both sides of (4), we obtain

$$\frac{d\bar{x}_n}{dm} = \left. \frac{-\log\left(\Lambda_\delta(x)/\lambda_\delta(x)\right) - \{S'(p_n(m))p_n(m)(1-p_n(m))n\}^{-1}}{(n-m)\lambda'_\delta(x)/\lambda_\delta(x) + (m-1)\Lambda'_\delta(x)/\Lambda_\delta(x) + \ell'_\delta(x)/\ell_\delta(x)} \right|_{x=\bar{x}_n(m)}. \quad (5)$$

The denominator is strictly negative, since $\lambda'_\delta < 0$, $\Lambda'_\delta < 0$, and $\ell'_\delta < 0$. In the numerator, the first term is strictly positive, since $\lambda_\delta(x)/\Lambda_\delta(x) > 1$ for any $x \in \Sigma$ by (1). The second term in the numerator is negative. However, the term converges to 0 as $n \to \infty$, because the derivative of the supply function is bounded and $0.5 \leq p_n(m) \leq \bar{p} < 1$ for any m and n. Therefore, for all sufficiently large values of n, we obtain that $d\bar{x}_n/dm < 0$.

Since $D(p_n(m))$ is the number of traders with $x_{\delta,i} \geq \bar{x}_n(m)$ for $m = 1, 2, \ldots, n$, divided by n, and since $D(p_n(0)) = D(p_n(1))$, the decreasing $\bar{x}_n(\cdot)$ implies that $D(p_n(m))$ is non-decreasing in m for any realization of $(x_{\delta,i})_{i=1}^n$. □

Lemma 1 indicates the presence of strategic complementarity in informed traders' buying decisions, as a higher price indicates that there are more informed traders who receive high signals. The mechanism in which demand feeds on itself is reminiscent of Bulow and Klemperer [8]'s "rational frenzies." Our model differs in that traders with private signals only observe an equilibrium price, whereas in their model, traders observe a series of auction prices through which private valuations are revealed sequentially.

The number of informed traders n needs to be large in order to obtain the upward sloping demand curve in our model. When n is small, the increment in price $p_n(m+1)/p_n(m)$ caused by an increase in demand becomes substantial due to limited supply, thus leading to a higher purchasing cost. For a small enough n, this increased purchasing cost overwhelms the effect of signal revealed by the increase in demand, leading to a downward sloping demand curve. Thus, in what follows we concentrate on the case where n is greater than n_o.

With the upward sloping demand function, we obtain the existence of equilibrium in a finite economy as follows.

Proposition 1 *For any $n > n_o$, there exists an equilibrium outcome (p_n^*, m_n^*) for each realization of $(x_{\delta,i})_{i=1}^n$.*

Proof: We define the aggregate reaction function as a mapping from the number of buying traders m to the number of buying traders determined by traders' choices given $p_n(m)$ and their private signals. Specifically, the aggregate reaction function is given by $\Gamma : \mathcal{M} \mapsto \mathcal{M}$, where $\mathcal{M} = \{0, 1, 2, \ldots, n\}$, for each realization of $(x_{\delta,i})_{i=1}^n$ such that $\Gamma(m) \equiv D(p_n(m))n$ for any $m \in \mathcal{M}$. Since Γ is a non-decreasing mapping of a finite discrete set $\mathcal{M}$ onto itself, there exists a non-empty closed set of fixed points of Γ as implied by Tarski's fixed point theorem. Since $S(p_n(m)) = m/n$, a fixed point m^* of Γ satisfies $D(p_n(m^*)) = S(p_n(m^*))$. Thus, we establish an equilibrium price as $p_n^* = p_n(m^*)$. $\square$

In this economy, multiple equilibria may exist for each realization of $(x_{\delta,i})_{i=1}^n$. We focus on the case where the auctioneer selects the minimum number of buying traders, m_n^*, among possible equilibria for each $(x_{\delta,i})_{i=1}^n$. This assumption that the auctioneer selects the minimum number of buying traders means that we exclude fluctuations that arise purely from informational coordination such as in sunspot equilibria. Even with this assumption, we can show that the equilibrium price, p_n^*, shows large fluctuations. Note that this equilibrium selection uniquely maps each realization of $(x_{\delta,i})_{i=1}^n$ to m_n^*. Thus, M_n^* is a random variable whose probability distribution is determined by the probability distribution of $(X_{\delta,i})_{i=1}^n$ and the equilibrium selection mapping.

3 Analytical derivation of the power law

In this section, we characterize the minimum equilibrium aggregate trading volume M_n^* and show that it follows a power law distribution. The power law distribution for M_n^* implies a fat tail and large volatility for the trading volume. Since the asset price in this model is determined by the equilibrium condition $S(p_n^*) = m_n^*/n$, the power law for the trading volume also implies a fat-tailed distribution of the equilibrium price P_n^*.

Let us consider a counting process $Y_o(x) \equiv \sum_{i=1}^{n} I_{X_{\delta,i} \geq x}$, where I is an indicator function: $I = 1$ if $X_{\delta,i} \geq x$ and $I = 0$ otherwise. $X_{\delta,i}$ follows a probability density function f_δ^H. Lemma 1 states that the threshold $\bar{x}_n(m)$ is decreasing in m. Thus, $Y_o(\bar{x}_n(m))$ is the number of buying traders at price $p_n(m)$, and $Y_o(\bar{x}_n(m))$ is non-decreasing in m. Then, equilibrium outcome m_n^* given signal profile $(x_{\delta,i})_{i=1}^n$ is equivalent to the minimum m such that $Y_o(\bar{x}_n(m)) = m$ given $(x_{\delta,i})_{i=1}^n$.

Define a change of variable as $t = \bar{x}_n^{-1}(x) - 1$. Note that t corresponds to $m - 1$ for $t = 0, 1, \ldots, n-1$. Since $\bar{x}_n(m)$ is monotone in m for sufficiently large n, $\tilde{f}_{\delta,n}(t) \equiv f_\delta^H(\bar{x}_n(t+1))|\bar{x}_n'(t+1)|$ is a probability density function of a signal defined over t. Now we transform the counting process $Y_o(x)$ to $Y(t)$, satisfying $Y_o(x = \bar{x}_n(m)) = Y(t = m-1)$ for $m = 1, 2, \ldots, n$. Then, M_n^* can be regarded as the first passage time for $Y(t) = t$.

When t increases from t to $t + dt$, a trader who chooses to buy before t continues to buy at $t+dt$, whereas a trader who chooses not to buy before t might switch to buying at $t+dt$. The conditional probability of a non-buying trader switching to buying between t and $t+dt$ for a small dt is equal to $q_{\delta,n}(t)dt \equiv \tilde{f}_{\delta,n}(t)dt/F_\delta^H(\bar{x}_n(t+1))$. Thus, the number of traders who buy between t and $t+dt$ for the first time, conditional on $Y(t)$, follows a binomial distribution with population parameter $n - Y(t)$ and probability parameter $q_{\delta,n}(t)dt$. The distribution of $Y(0)$ follows a binomial distribution with population n

and probability $q^o_{\delta,n} \equiv 1 - F^H_\delta(\bar{x}_n(1))$. This completes the definition of the stochastic process $Y(t)$ for $t \geq 0$.

Let $\phi_{\delta,n}(t)dt$ denote the mean of $Y(t+dt) - Y(t)$ for a small dt. Thus, $\phi_{\delta,n}(t) \equiv q_{\delta,n}(t)(n - Y(t))$. For a finite $Y(t)$, $Y(t+dt) - Y(t)$ asymptotically follows a Poisson distribution with mean $\phi_{\delta,n}(t)$ as $n \to \infty$. Hence, for sufficiently large n, $Y(t)$ asymptotically follows a Poisson process with time-dependent intensity $\phi_{\delta,n}(t)$. The following lemma characterizes an asymptotic behavior of the intensity function $\phi_{\delta,n}$ as $n \to \infty$.

Lemma 2 *As $n \to \infty$, $Y(t)$ asymptotically follows a Poisson process with intensity:*

$$\lim_{n\to\infty} \frac{\log \ell_\delta(\bar{x}_n(t+1))}{\ell_\delta(\bar{x}_n(t+1)) - 1}. \tag{6}$$

Proof: First, we show that $\bar{x}_n(t) \to \bar{\Sigma}$ as $n \to \infty$. Equation (4) is rewritten as:

$$\lambda_\delta(\bar{x}_n(t))^n = \left(\frac{1}{p_n(t)} - 1\right)\left(\frac{\lambda_\delta(\bar{x}_n(t))}{\Lambda_\delta(\bar{x}_n(t))}\right)^t \frac{\Lambda_\delta(\bar{x}_n(t))}{\ell_\delta(\bar{x}_n(t))}.$$

The right-hand side is bounded for any n. Hence, $\lambda_\delta(\bar{x}_n(t)) \to 1$ as $n \to \infty$. This implies that $\bar{x}_n(t) \to \bar{\Sigma}$ as $n \to \infty$. Next, we transform $\phi_{\delta,\infty} \equiv \mathrm{plim}_{n\to\infty} \phi_{\delta,n}$ using (2), (3), and (5).

$$\begin{aligned}
\phi_{\delta,\infty}(t) &= \operatorname*{plim}_{n\to\infty} \frac{\tilde{f}_{\delta,n}(t)}{F^H_\delta(\bar{x}_n(t+1))}(n - Y(t)) = \operatorname*{plim}_{n\to\infty} \frac{f^H_\delta(\bar{x}_n(t+1))|\bar{x}'_n(t+1)|}{F^H_\delta(\bar{x}_n(t+1))}(n - Y(t)) \\
&= \operatorname*{plim}_{n\to\infty}(n - Y(t))\frac{f^H_\delta(x)}{F^H_\delta(x)} \left.\frac{\log\left(\Lambda_\delta(x)/\lambda_\delta(x)\right) + \{S'(p_n(t+1))p_n(t+1)(1-p_n(t+1))n\}^{-1}}{(n-t-1)\lambda'_\delta(x)/\lambda_\delta(x) + t\Lambda'_\delta(x)/\Lambda_\delta(x) + \ell'_\delta(x)/\ell_\delta(x)}\right|_{x=\bar{x}_n(t+1)} \\
&= \operatorname*{plim}_{n\to\infty}\left(1 - \frac{Y(t)}{n}\right) \left.\frac{\log(\Lambda_\delta(x)/\lambda_\delta(x)) + \{S'(p_n(t+1))p_n(t+1)(1-p_n(t+1))n\}^{-1}}{\left(1 - \frac{t+1}{n}\right)\left(\frac{\ell_\delta(x)}{\lambda_\delta(x)} - 1\right) + \frac{F^H_\delta(x)}{nf^H_\delta(x)}\left(\frac{t\Lambda'_\delta(x)}{\Lambda_\delta(x)} + \frac{\ell'_\delta(x)}{\ell_\delta(x)}\right)}\right|_{x=\bar{x}_n(t+1)}
\end{aligned} \tag{7}$$

We examine the large fraction in (7). Note that ℓ_δ and ℓ'_δ are bounded. Since $\Lambda_\delta(x) \sim \ell_\delta(x)$ as $x \to \bar{\Sigma}$, and since $\bar{x}_n(t+1) \to \bar{\Sigma}$ as $n \to \infty$, $\lim_{n\to\infty} \Lambda'_\delta(\bar{x}_n(t+1))$

is also bounded. Moreover, $f_\delta^H(x)$ is strictly positive for any $x \in \Sigma$. Thus, the second term in the denominator of (7) vanishes as $n \to \infty$.

The second term in the numerator of (7) also converges to 0 as $n \to \infty$, since S' is bounded and $p_n \in [0.5, \bar{p}]$. Finally, $\log(\Lambda_\delta(x)/\lambda_\delta(x))$ in the first term in the numerator of (7) and $\ell_\delta(x)/\lambda_\delta(x) - 1$ in the first term in the denominator of (7) are bounded away from zero. Thus, $\phi_{\delta,\infty}(t)$ is bounded. This implies that the asymptotic variance of $Y(t+dt) - Y(t)$, $\lim_{n\to\infty}(n - Y(t))(1 - q_{\delta,n}(t)dt)q_{\delta,n}(t)dt$, is also bounded. Hence as $n \to \infty$, $Y(t)/n$ converges in the L^2-norm, and thus in probability, to 0.

Applying this result to (7), we obtain that:

$$\phi_{\delta,\infty}(t) = \operatorname*{plim}_{n\to\infty} \frac{\log \Lambda_\delta(\bar{x}_n(t+1)) - \log \lambda_\delta(\bar{x}_n(t+1))}{\ell_\delta(\bar{x}_n(t+1))/\lambda_\delta(\bar{x}_n(t+1)) - 1}.$$

Noting that $\bar{x}_n(t+1) \to \bar{\Sigma}$ as $n \to \infty$ as well as that $\Lambda(x) \sim \ell(x)$ and $\lambda(x) \to 1$ as $x \to \bar{\Sigma}$, we obtain the expression (6) for $\phi_{\delta,\infty}(t)$. □

The likelihood function $\ell_\delta(\cdot)$ is restricted by $\sup_{x\in\Sigma} |\ell_\delta(x) - 1| < \delta$. It is evident from L'Hôpital's rule that $(\log \ell_\delta(x))/(\ell_\delta(x) - 1)$ uniformly converges to 1 as $\delta \to 0$. Thus, $Y(t)$ asymptotically follows the Poisson process with intensity 1 as $\delta \to 0$. Using this, we will show that the first passage time of $Y(t)$ converges in distribution to that of the Poisson process with intensity 1 as $\delta \to 0$.

We focus on the first passage time conditional on $Y(0) = c$ for some positive integer c. The initial condition implies $Y_o(\bar{x}_n(1)) = c$, namely that there are c traders who receive private information greater than $\bar{x}_n(1)$. As $n \to \infty$, $Y(t)$ asymptotically follows a Poisson process with intensity $\phi_{\delta,n}(t)$, which starts at $Y(0) = c$. Let $\tau_{\phi_{\delta,n}(\cdot)}$ denote the first passage time of $Y(t)$ reaching t. Then, $\tau_{\phi_{\delta,n}(\cdot)}$ is also the first passage time of $Y(t) - Y(0)$ reaching $t - c$. Let us define $N(t)$ as the Poisson process with constant intensity 1 and $N(0) = 0$. Then, τ_1 denotes the first passage time of $N(t)$ reaching $t - c$. An inhomogeneous Poisson process with intensity $\phi_{\delta,n}(t)$ can be transformed by a change of time to a homogeneous Poisson process as $N(\int_0^t \phi_{\delta,n}(s)ds)$. Thus, the first passage time we consider is

$$\tau_{\phi_{\delta,n}(\cdot)} \equiv \inf\left\{t \geq 0 \mid N\left(\int_0^t \phi_{\delta,n}(s)ds\right) \leq t - c\right\}$$

where $\inf \emptyset \equiv \infty$.

We consider a vanishingly small δ, which restricts $\sup_{x\in\Sigma} |\ell_\delta(x) - 1|$. The small δ implies that signal $X_{\delta,i}$ is close to a pure noise. This case occurs, for example, in a very high frequency trading in which the information content traders obtain from signals during a trading period is quite small. With this setup, the following lemma establishes that the first passage time of the inhomogeneous Poisson process $Y(t)$ converges in distribution to the first passage time of the standard Poisson process $N(t)$.

Lemma 3 *As $n \to \infty$ and $\delta \to 0$ simultaneously, $\tau_{\phi_{\delta,n}(\cdot)}$ converges in distribution to τ_1.*

Proof: Since $\sup_{x\in\Sigma} |\ell_\delta(x) - 1| < \delta$, $\ell_\delta(\cdot)$ uniformly converges to 1 as $\delta \to 0$. Also, $1 \leq (\log \ell_\delta(x))/(\ell_\delta(x) - 1) < -(\log(1-\delta))/\delta$. Thus, applying L'Hôpital's rule, $(\log \ell_\delta(x))/(\ell_\delta(x) - 1)$ converges to 1 as $\delta \to 0$.

Under this setup, we show that the random variable $\tau_{\phi_{\delta,n}(\cdot)}$ defined over $[0,\infty]$ converges in distribution to τ_1 as a pair (n, δ) tends to $(\infty, 0)$. We prove this by showing that the Laplace transform of $\tau_{\phi_{\delta,n}(\cdot)}$ converges to that of τ_1 as $n \to \infty$ and $\delta \to 0$ simultaneously. In (7), we observe that $\phi_{\delta,n}(t)$ contains a stochastic term $Y(t)/n$, which converges in probability to 0 as $n \to \infty$. That is, for a constant y, the probability for the events $Y(t)/n > y$ becomes arbitrarily small for large n. Thus, the contribution of such events to the Laplace transform of $\tau_{\phi_{\delta,n}(\cdot)}$ is arbitrarily small for large n. Moreover, the intensity $\phi_{\delta,n}(t)$ can be set arbitrarily close to $(\log \ell_\delta(\bar{x}_n(t+1)))/(\ell_\delta(\bar{x}_n(t+1)) - 1)$ for sufficiently large n. Thus, $1 \leq \phi_{\delta,n}(t) < -(\log(1-\delta))/\delta$ for large n. Hence for large

n, $\phi_{\delta,n}(t)$ can be set arbitrarily close to 1 as $\delta \to 0$. Therefore, it is sufficient to show that for any $\beta > 0$,

$$\lim_{\delta \to 0} \mathbb{E}\left[\exp(-\beta \tau_{\phi_{\delta,n}(\cdot)})\right] = \mathbb{E}\left[\exp(-\beta \tau_1)\right]. \tag{8}$$

Note that $e^{-\beta\tau}$ is set at 0 for the events where $\tau = \infty$ by convention.

Since an inhomogeneous Poisson process can be transformed to a homogeneous Poisson process with a change of time, inequalities $\tau_1 \leq \tau_{\phi_{\delta,n}(\cdot)} \leq \tau_{-(\log(1-\delta))/\delta}$ hold. Thus, in order to establish (8), it is sufficient to show that $\mathbb{E}[\exp(-\beta\tau_\psi)]$ is continuous with respect to ψ. We also note that

$$\begin{aligned}
\tau_\psi &= \inf\{t \geq 0 \mid N(\psi t) \leq t - c\} \\
&= \inf\{t \geq 0 \mid t - N(\psi t) \geq c\} \\
&= \frac{1}{\psi} \inf\left\{t \geq 0 \mid \frac{t}{\psi} - N(t) \geq c\right\} \\
&= \frac{1}{\psi}\tilde{\tau}_\psi
\end{aligned}$$

where $\tilde{\tau}_\psi \equiv \inf\{t \geq 0 \mid t/\psi - N(t) \geq c\}$.

Let ζ be a constant in $(0,1)$. Consider a stochastic differential equation:

$$dZ(t) = -\zeta Z(t-)\{dN(t) - dt\}, \quad Z(0) = 1.$$

The solution of the stochastic differential equation is a martingale and satisfies

$$Z(t) = e^{\zeta t}(1-\zeta)^{N(t)} = \left(\frac{1}{1-\zeta}\right)^{\frac{t}{\psi} - N(t)} \exp\left\{\left(\zeta + \frac{\log(1-\zeta)}{\psi}\right)t\right\}.$$

Now, for fixed β and ψ, there exists a unique ζ that satisfies an equation

$$\zeta\psi + \log(1-\zeta) = -\beta.$$

Let $\zeta(\beta, \psi)$ denote the unique solution. Note that $\zeta(\beta, \psi)$ is continuous and monotonically increasing with respect to both β and ψ. Then, Z is written as

$$Z(t) = \left(\frac{1}{1-\zeta(\beta,\psi)}\right)^{\frac{t}{\psi}-N(t)} \exp\left(-\frac{\beta}{\psi}t\right).$$

$Z(t)$ is positive and takes a value less than or equal to $\{1-\zeta(\beta,\psi)\}^{-c}$ at and before the stopping time $\tilde{\tau}_\psi$. Namely, $Z(t)$ is bounded. Therefore, $\mathbb{E}[Z(\tilde{\tau}_\psi)] = 1$ holds by the optional sampling theorem. (Note that $Z = 0$ for the events where $\tilde{\tau}_\psi = \infty$.) Moreover, noting that $N(t)$ does not jump at the point of time $\tilde{\tau}_\psi$, we obtain that

$$Z(\tilde{\tau}_\psi) = \left(\frac{1}{1-\zeta(\beta,\psi)}\right)^{c} \exp\left(-\frac{\beta}{\psi}\tilde{\tau}_\psi\right),$$

for both cases of $\tilde{\tau}_\psi < \infty$ and $\tilde{\tau}_\psi = \infty$. Thus,

$$\mathbb{E}[\exp(-\beta\tau_\psi)] = \mathbb{E}\left[\exp\left(-\frac{\beta}{\psi}\tilde{\tau}_\psi\right)\right] = \{1-\zeta(\beta,\psi)\}^{c}.$$

Since $\zeta(\beta, \psi)$ is continuous with respect to ψ, this completes the proof. □

We had shown that the equilibrium number of buying traders M_n^* has the same distribution as the stopping time $\tau_{\phi_{\delta,n}}$ of a counting process $Y(t)$. Lemma 2 showed that $Y(t)$ asymptotically follows a Poisson process as $n \to \infty$. Lemma 3 then shows that $\tau_{\phi_{\delta,n}}$ converges in distribution to τ_1 for large n as $\delta \to 0$, i.e., when a large number of traders receive signals that contain little information.

We can further derive the distribution function of τ_1 explicitly, using the fact that τ_1 has the same distribution function as the sum of a branching process. The stopping time τ_1 follows the same distribution as M_n^* conditional on that there are $Y(0) = c$ traders who receive private information $x_{\delta,i} \geq \bar{x}_n(1)$. Hence, we obtain the conditional distribution of the equilibrium number of buying traders, $M_n^* \mid Y(0)$, for sufficiently large n and small δ.

Proposition 2 *As* $n \to \infty$ *and* $\delta \to 0$ *simultaneously,* M_n^* *conditional on* $Y(0) = c$ *asymptotically follows:*

$$\Pr\left(M_n^* = m \mid Y(0) = c\right) = (c/m)e^{-m}m^{m-c}/(m-c)!, \quad m = c, c+1, \ldots$$

Moreover, the tail of the asymptotic distribution follows a power law with exponent 0.5, i.e., $\Pr(M_n^* > m) \propto m^{-0.5}$ *for sufficiently large values of* m*.*

Proof: Consider the Poisson process $N(t)$ with intensity 1 and $N(0) = 0$. The first passage time τ_1 of $N(t)$ reaching $t - c$ must be greater than or equal to c. Now we introduce a process b with $b(0) = c$. During the time interval c, the increment $N(c) - N(0)$, denoted as $b(1)$, follows a Poisson distribution with mean c. Since a Poisson random variable is infinitely divisible, a Poisson random variable with mean c is equivalent to c-fold convolution of the Poisson with mean 1. Thus, we can regard $b(1)$ as the sum of "children" borne by $c = b(0)$ "parents," where each parent bears a number of children following the Poisson with mean 1. If $b(1) = 0$, the process b stops, and the first passage time is $b(0) = c$. If $b(1) > 0$, the first passage time is greater than or equal to $b(0) + b(1)$. During the time interval $(b(0), b(0) + b(1)]$, new increment $b(2) \equiv N(b(0) + b(1)) - N(b(0))$ follows the Poisson distribution with mean $b(1)$, which is equivalent to $b(1)$-fold convolution of the Poisson with mean 1 and regarded as the number of children borne by $b(1)$ parents (note that the increment $b(1)$ of a Poisson process is always an integer). This process $b(u)$ continues for $u = 1, 2, \ldots, U$, where U denotes the stopping time at which $b(U)$ is equal to 0 for the first time. Thus, the first passage time τ_1 is equal to $\sum_{u=0}^{U} b(u)$, the total number of population generated in the so-called Poisson branching process $b(u)$ in which each parent bears a number of children according to the Poisson distribution with mean 1.

It is known that the sum of the Poisson branching process, cumulated over time until the process stops, follows a Borel-Tanner distribution (Kingman [19]). When the Poisson mean of the branching process $b(u)$ is $\phi > 0$ generally, the Borel-Tanner distribution is written as:

$$\Pr\left(\sum_{u=0}^{U} b(u) = m \mid b(0) = c\right) = \frac{c}{m}\frac{e^{-\phi m}(\phi m)^{m-c}}{(m-c)!}, \quad m = c, c+1, \dots \tag{9}$$

$$\propto \; e^{-(\phi-1-\log\phi)m} m^{-1.5}, \quad \text{as } m \to \infty. \tag{10}$$

The tail characterization in (10) is obtained by applying Stirling's formula to (9). Since τ_1 follows the same distribution as the sum of the Poisson branching process with mean 1, it follows (9) and (10) with $\phi = 1$. □

Proposition 2 shows that the distribution of M_n^* conditional on $Y(0) = c$ has a power-law tail. This implies that, given there are c traders who receive extremely favorable private signals, their buying actions may trigger a stochastic herd, and the size of the herd follows a fat-tailed distribution. We can pin down the distribution of $Y(0)$ under a certain condition, in which case we can explicitly derive an unconditional asymptotic distribution of M_n^*. For finite n, $Y(0)$ follows a binomial distribution with population n and probability $q_{\delta,n}^o \equiv 1 - F_\delta^H(\bar{x}_n(1))$. The behavior of the asymptotic mean $\phi_\delta^o \equiv \lim_{n\to\infty} nq_{\delta,n}^o$ depends on specification of signals $X_{\delta,i}$. If the asymptotic mean is finite, $Y(0)$ follows a Poisson distribution with mean ϕ_δ^o asymptotically as $n \to \infty$. We obtain the unconditional distribution explicitly in this case as follows.

Proposition 3 *Suppose that $n(1 - F_\delta^H(\bar{x}_n(1)))$ converges to a positive constant ϕ_δ^o as $n \to \infty$. Then for sufficiently small δ and large n, the distribution function of M_n^* is arbitrarily close to:*

$$\Pr(M_n^* = m) = \frac{\phi_\delta^o e^{-m-\phi_\delta^o}}{m!} (m + \phi_\delta^o)^{m-1}, \quad m = 0, 1, \dots. \tag{11}$$

Moreover, M_n^* has a power-law tail distribution with exponent 0.5.

Proof: The unconditional distribution is derived by combining the distribution (9) with $\phi = 1$ and the Poisson distribution with mean ϕ^o_δ for $Y(0)$ and by using the binomial theorem as follows.

$$\begin{aligned}\sum_{c=0}^{m} \Pr\left(M^* = m \mid Y(0) = c\right) \Pr(Y(0) = c) &= \sum_{c=0}^{m} \frac{c}{m} \frac{e^{-m} m^{m-c}}{(m-c)!} \frac{(\phi^o_\delta)^c}{c!} e^{-\phi^o_\delta} \\ &= \frac{\phi^o_\delta e^{-m-\phi^o_\delta}}{m!} \sum_{c=0}^{m} \frac{(m-1)!}{(m-c)!(c-1)!} m^{m-c} (\phi^o_\delta)^{c-1} \\ &= \frac{\phi^o_\delta e^{-m-\phi^o_\delta}}{m!} (m + \phi^o_\delta)^{m-1}, \quad m = 0, 1, \ldots .\end{aligned}$$

Thus, we obtain (11). Applying Stirling's formula, we obtain that the tail follows a power law with exponent 0.5. □

Propositions 2 and 3 indicate the presence of a power-law tail for the equilibrium number of buying traders. The power law implies a large variance of M^*_n. In general, a power law with exponent α implies that any k-th moment for $k \geq \alpha$ is infinite. Thus, with exponent 0.5, M^*_n does not have a finite variance or mean.

The power law also implies that the variance of the fraction of buying traders, M^*_n/n, can be quite large. Note that M^*_n is bounded by n in a market with a finite number of informed traders. By integrating $(M^*_n/n)^2$ up to $M^*_n = n$ with a power-law tail exponent 0.5, we find that the variance of M^*/n decreases as $n^{-0.5}$ when n becomes large. This contrasts with the case when the traders act independently. If traders' choices $(d_{n,i})_{i=1}^n$ were independent, the central limit theorem predicts that M^*_n/n would asymptotically follow a normal distribution, whose tail is thin and variance declines as fast as n^{-1}. The variance of M^*/n differs by factor $\sqrt{n}$ between our model and the model with independent choices. This signifies the effect of stochastic herding that magnifies the small fluctuations in the average of signals $X_{\delta,i}$. Even though a magnification effect occurs whenever traders' actions are correlated, it requires a particular structure in correlation among traders for the magnification effect to cause the variance to decline more slowly than n^{-1}. The magnification effect in our model is analogous to a long memory process, in which a large deviation from the long-run mean is caused by long-range autocorrelation. In our static model, the long-range correlation of traders' actions is captured by the asymptotic martingale process $Y(t)$ when $\delta \to 0$. The power law exponent 0.5 obtained in our model is closely related to the same exponent in the Inverse Gaussian distribution that characterizes the first passage time of a martingale. An economic meaning of $Y(t)$ being a martingale in our model is that the mean number of traders induced to buy by a buying trader is 1. We will argue that such an environment is analogous to the indeterminate equilibrium that occurs in Keynes's beauty contest, in which the average action of a single trader responds one-to-one to the average actions of the entire group.

A power law of M_n^* implies that equilibrium trading volumes can occur at any order of magnitude. This wide range of equilibrium aggregate outcomes can be seen in the economic environment that gives rise to indeterminacy of equilibria. Indeterminacy in signal inference games is best exemplified by Keynes's beauty contest, in which voters care more about who is selected by other voters rather than who is actually beautiful, such that any candidate can win, regardless of inherent quality. A property of our model, similar to that of Keynes's beauty contest, can be seen from optimal threshold condition (4). This condition reduces to the simple form $(1-\mu)\log\lambda_n(\bar{x})+\mu\log\Lambda_n(\bar{x}) = 0$, where $\mu \equiv m/n$, if we take the limit as n approaches infinity while keeping μ unchanged. The condition indicates that the log of the geometric average of λ and Λ evaluated at $\bar{x}$, which can be regarded as a summary statistic for information on the true state revealed by traders' actions, does not change even when μ takes different values in equilibrium.

To explain why this happens, suppose that a trader switches from not-buying to buying. This increases μ, which leads to a decline in the geometric average of λ and Λ that traders observe, such that the optimal threshold declines. This in turn increases the average likelihood ratio, because each trader learns that the signals received by non-buying traders must have been below the threshold level. As a result, the impact of a change in μ on the geometric average of λ and Λ is exactly canceled out, which makes it possible that any value of μ satisfies the above reduced condition. The setup of our model, with a finite number of traders and discrete actions, prevents this type of local indeterminacy from occurring. However, the indeterminacy described above is useful to understand the key environment of our model that generates the large fluctuations of the aggregate trading volume.

It is important to note that the ability of the above mechanism to generate indeterminacy depends on the information structure adopted. Specifically, if there exists substantial heterogeneity in the information structure as to who observes whose actions, a trader observed by many traders would provide a stronger herding trigger. A useful example is Banerjee's sequential herding model, in which agents observe only the actions of those agents who move before them. Under this information structure, it is possible that the first mover's action cascades to all agents, with private signals of most agents remaining unrevealed. An important implication of Banerjee's model is that intermediate outcomes between "herding" and "no herding" do not occur. This contrasts sharply with our result that any degree of herding can be realized in equilibrium. The difference arises from the information structure, which is assumed symmetric across traders in our model.

Propositions 2 and 3 claim not only that various levels of aggregate trading volumes M_n^* are possible, but also that the distribution of M_n^* has a particular regularity signified by a power law. The power law for M_n^* implies a particular fat-tailed distribution of the equilibrium price P_n^*. In the next section, we explore through numerical analysis how the particular distribution of the price generated by our model can account for stock price movements observed in reality.

4 Numerical results on volume and return distributions

In this section, we conduct numerical simulations of the model with a finite number of informed traders n. The purpose of this simulation exercise is to numerically show that the probability distribution of the number of buying traders M_n^* has a power law tail, which was shown in the previous section as an asymptotic property when n tends to infinity. Moreover, we will look at fluctuations in equilibrium asset returns $\log P_n^* - \log p(0)$ to make sure that the return distribution exhibits a fat tail, and it matches well with the return distribution observed in actual data.

An important facet of the model to be specified is the supply function $S(p)$, which determines how the fluctuation of volumes is translated to the fluctuation of returns. In our model, where informed traders' demands are absorbed by uninformed traders' supply, the elasticity of supply function determines the impact of demand shifts on the returns. The relation between an exogenous shift in trading volume and a resulting shift in asset price is often called a price impact function. We adopt a square-root specification of the price impact function. Namely, we specify the inverse supply schedule of uninformed traders as $p(m) = p(0) + p(0)(m/n)^{\gamma}$, for $m = 1, 2, \ldots, n$, with $\gamma = 0.5$. A micro-foundation for the square-root specification is provided by Gabaix et al. [13] in a Barra model of uninformed traders who have a mean-variance preference and zero bargaining power against informed traders. The square-root specification is commonly used for the price impact (e.g., Hasbrouck and Seppi [16]), and its parameter specification, $\gamma = 0.5$, falls within the empirically identified range of the price impact by Lillo et al. [21].

Other parts of the model are specified as follows. The distributions of signal, F^s, are specified as normal distributions. The mean of F^H and F^L are set at $\mu_H = 1$ and $\mu_L = 0$, respectively. F^H and F^L have a common standard deviation σ. We set σ at 25 or 50. This large standard deviation relative to the difference in mean of one captures the situation where the informativeness of signal X_i is small. We set the number of informed traders n at a finite but large value between 500 and 4000. Under these parameter values, the optimal threshold function $\bar{x}_n(\cdot)$ is computed. Using the threshold function, we conduct Monte Carlo simulations. A profile of private signals $(x_i)_{i=1}^n$ is randomly drawn 100,000 times, and m_n^* and p_n^* are computed for each draw.

Figure 1 plots the complementary cumulative distribution of M_n^*/n for various parameter values of n and σ. The complementary distribution $\Pr(M_n^* > m)$ is cumulated from above, and is thus 0 at $m = n$. The distribution is plotted in log-log scale. Thus, a linear line indicates a power law $\Pr(M_n^* > m) \propto m^{-\alpha}$, where the slope of the linear line α is the exponent of the power law. The simulated distributions appear linear for a wide range of M_n^*. This conforms to the model prediction that M_n^* follows a power law distribution. The simulated distribution decays fast when M_n^*/n is close to 1, due to the finiteness of n.

The asymptotic results in Propositions 2 and 3 predicted the exponent of power law α to be 0.5. As shown in the left panel of Figure 1, we observe that the power law exponent of the simulated M_n^* is roughly equal to 0.5 when $n = 1000$ and $\sigma = 25$. We check the robustness of the result by simulating the model with a larger number of

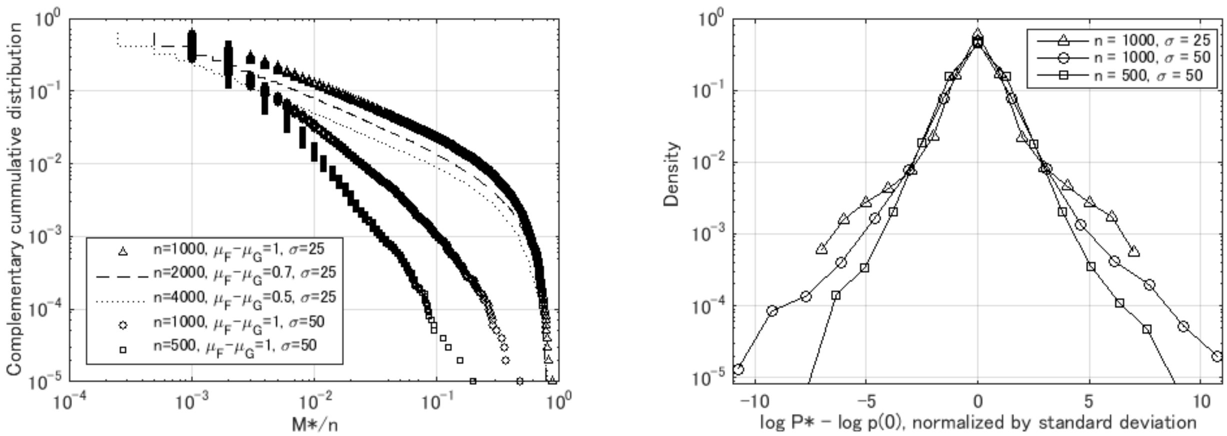

Figure 1: Left: Simulated complementary cumulative distributions of the minimum equilibrium number of buying traders M_n^*. n is the number of traders, σ is the standard deviation of the private information, and $\mu_H - \mu_L$ denotes the difference of the mean of the private information between F^H and F^L. Right: Simulated distributions of returns $\log P_n^* - \log p(0)$.

traders and lower signal informativeness. In order to set a lower level of informativeness, we lower the mean difference $\mu_H - \mu_L$ from 1 to 0.7 or 0.5 with σ fixed. In Figure 1, we observe that a similar slope $\alpha = 0.5$ for the power law holds for the cases with $n = 2000$ (dashed line) and 4000 (dotted line) when the informativeness of signal is lower. In the simulations under other parameter sets, however, we note that α can take larger values. This can be seen in the plot for a larger σ (circle-line) and a smaller n (square-line). This deviation in the exponent might result from the fact that the finite truncation occurs at a relatively small value of m_n^* in these cases. It is also possible that the state-dependence of the intensity $\phi_{\delta,n}$ is strong enough to cause a large deviation from the predicted exponent $\alpha = 0.5$. Sornette [30] showed that in these types of "criticality" models, the power law exponent increases by 1 when the parameter $\phi_{\delta,n}$ fluctuates around the criticality value, 1.

Our model also determines price p_n^* for each equilibrium number of buying traders m_n^*. We interpret the shifts in log price, $\log p(m_n^*) - \log p(0)$, as stock returns. To investigate the fit of the model to observed distributions of stock returns, it must be extended such that informed traders herd on the sell side as well as on the buy side. Here, we simply assume that trading sessions alternate between two cases when the informed traders can buy and when they can sell.[5] When the informed traders herd on the supply side, m_n^* is interpreted as the number of selling traders, and $\log p(m_n^*) - \log p(0)$ is interpreted as an associated negative return. We plot the distributions of the simulated returns in the right panel of Figure 1. The density is logarithmically scaled, and thus, a linear decline indicates an exponential distribution. Note that the returns

[5] It would be more natural if we allow the informed traders to choose between buying, selling, and inaction simultaneously. We conjecture that our results will also hold in such an extended model. However, such an extension would lead to complications that involve various cases of $Y(0)$ (how many traders respond initially by buying or selling) without generating additional insights.

are normalized by the standard deviations of $\log P_n^* - \log p(0)$. The normalized returns still span a wide range from -10 to 10. Thus, the plots clearly indicate the presence of fat tails in the simulated returns distributions.

The simulated distribution of returns is compared to the observed distribution in Figure 2. The observed distribution is generated using daily returns data of TOPIX stock price index in the Tokyo Stock Exchange from 1998 to 2010. We define the daily return as the log difference from the opening price to the closing price. We use the opening-closing difference rather than the return in a business day in order to homogenize the time horizon of each observed return. The simulated distribution is generated under $n = 1000$. The standard deviation σ of the signal is set to 48.5, at which value the density estimate of simulated returns at 0 matches with that of the observed distribution. The other parameters are set as before: $\gamma = 0.5$, $\mu_H = 1$, and $\mu_L = 0$.

In the left panel of Figure 2, the returns distributions are plotted in semi-log scale. The plot shows that the simulated distribution traces the observed distribution rather well, especially in the left tail. In the same panel, we plot the standard normal density by a dotted line. Even though the simulated and observed distributions are normalized by their standard deviations, both distributions deviate substantialy from the normal distribution in the tails at more than three standard deviations away from the mean. Note that we used σ as a free parameter in the simulation to match the observed density at 0, but we did not use it to match the tail distribution. This indicates that our model is capable of generating the fat tail of observed returns better than models that generate the normal distribution.

To further investigate the match between the simulated and observed distributions, we show a Q-Q plot in the right panel of Figure 2. In the Q-Q plot, each quantile of

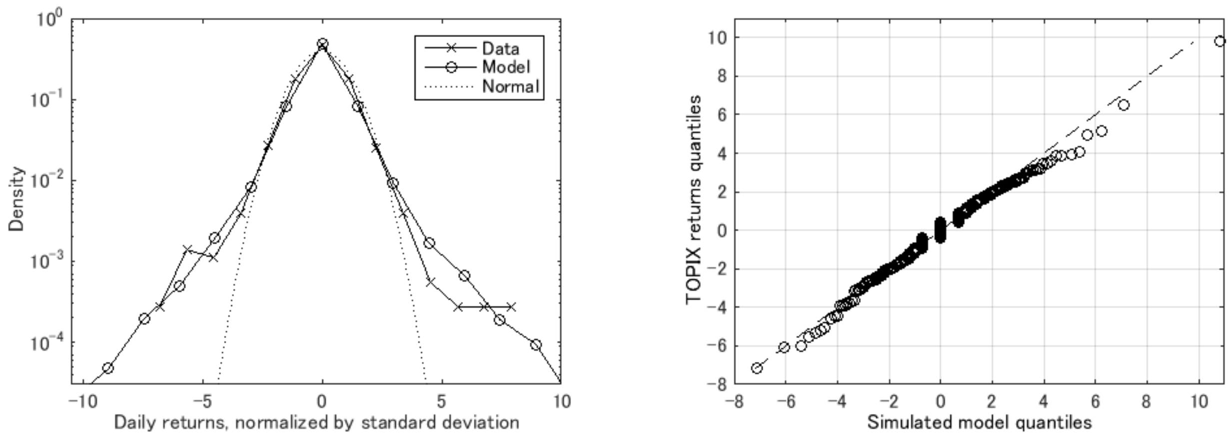

Figure 2: Distributions of TOPIX daily returns and simulated returns $\log P_n^* - \log p(0)$. *Left:* Distributions plotted in semi-log scale, where returns are normalized by standard deviations. Observed and model distributions are shown along with a standard normal distribution. *Right:* Quantile-quantile (Q-Q) plot. Each circle represents a pair of values, the simulated data value in the horizontal axis and the TOPIX data value in the vertical axis, under which the two distributions in comparison have the same fraction of the population.

the TOPIX returns data is plotted against the same quantile of the simulated returns data. Thus, the two distributions are identical if the Q-Q plot coincides with the 45 degree line, shown by a dashed line. Both quantiles are normalized by their standard deviations. In Figure 2, the two quantiles follow the 45 degree line reasonably closely overall, although the simulated quantiles somewhat overshoot the observed quantiles in the region greater than 2.

5 Conclusion

This study analyzed aggregate fluctuations of trading volumes and prices that arise from information inference behaviors among traders in financial markets. In a class of herd behavior models in which each trader infers the private information of other traders only by observing their actions, we found that the number of traders taking the same action at equilibrium exhibits large volatility with a statistical regularity—a power-law distribution. Furthermore, we showed that the model is capable of generating a fat-tailed distribution of asset returns. The simulated distribution of equilibrium returns was demonstrated to match well with the distribution of observed stock returns.

The power-law distribution of trading volumes emerges when the information structure of traders is symmetric. Every trader receives a private signal containing the same magnitude of informativeness regarding the true value of an asset, and every trader observes the average action of all traders. In our model, an action by one trader is as informative as inaction by another. When information is revealed by a trader's buying action, the inaction of other traders reveals their private information in favor of not buying. Thus, each trader's action is influenced by the average action, resulting in a near-indeterminate equilibrium analogous to Keynes's beauty contest. In this way, our information inference model provides an economic foundation for the criticality condition that generates power-law fluctuations.

This study suggests several directions for extension. The present static model is shown to match with the quantitative properties of unconditional fluctuations. The natural next step would be to develop a dynamic model that accounts for the time-series properties as pursued by, for example, Alfarano, et al. [1]. A dynamically extended model is presented in the working paper version of this study (Nirei [26]), which generates a time-series pattern similar to Lee [20] for sudden shifts in stock prices. Another direction would be to extend the model by incorporating more realistic market structures. Kamada and Miura [18] have taken a step in this direction by extending this model to the case where both public and private signals exist and where informed traders can take both buying and selling sides. We hope that our study provides a valuable step toward promoting subsequent research on fat-tailed distributions in financial markets.

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