Beauty Contests and Fat Tails in Financial Markets

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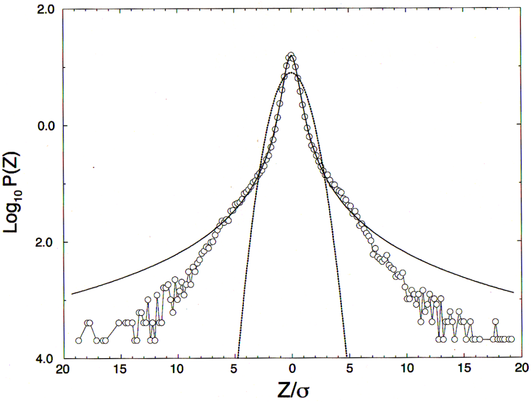
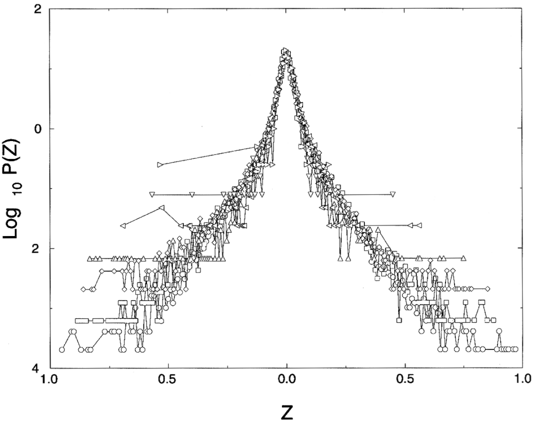
Empirical literature on fat tails in finance

- Stock returns follow a fat-tailed distribution
  - Evident in the high-frequency domain (Mandelbrot 1963; Fama 1963)
  - The tail regularity could span historical crashes (Jansen and de Vries, REStat 1991; Longin, JB 1996)
  - Leptokurtic (4th moment greater than the normal)
- Trading volumes also show a fat tail (Gopikrishnan, Plerou, Gabaix, and Stanley 2000)
  - "It takes volume to move prices"

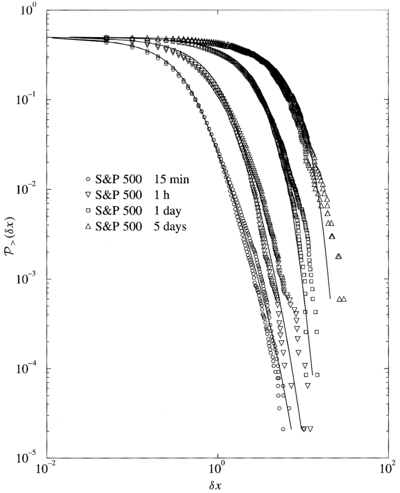
Fat tails of stock returns



S&P 500 index, 1 minute interval, 6 years coverage. Source: Mantegna and Stanley, 2000, Cambridge



Source: Mantegna and Stanley



Source: Bouchaud and Potters, 2000, Cambridge

Tail distributions

- Gaussian $\phi(x) \propto e^{-(x-\mu)^2/2\sigma^2}$
  - Parabola in a semi-log plot
- Exponential tail $\Pr(X > x) \propto e^{-\lambda x}$
  - Linear in a semi-log plot
- Power law tail $\Pr(X > x) \propto x^{-\alpha}$
  - Linear in a log-log plot
  - Does not have a finite variance if $\alpha < 2$
  - ...nor a finite mean if $\alpha \leq 1$ (e.g. Cauchy)

Tail matters

- Fat tail affects *risks*
  - volatility
  - option price
  - value at risk
- Power-law tail suggests the same mechanism for price fluctuations, small and large
  - fractal, self-similar, scale-free
  - crash
  - high frequency data

Plan of the paper

- Develop a simultaneous-move rational-herding model of securities traders with private signal
- Derive a distribution of equilibrium aggregate actions
- Match with an empirical fat-tailed distributions of stock trading volumes and returns
- Provide an economic reason why the fat tail has to occur

Signal

- Two states of the economy: H (High) and L (Low)
- True state is H.
- Common prior belief $\Pr(H) = \Pr(L) = 1/2$
- Each informed trader receives private signal $X_{\delta,i}$ i.i.d. across i, which follows cdf F_δ^s in state $s = H, L$ with common support Σ where $\sup \Sigma = \bar{\Sigma} < \infty$. Also $f_\delta^s(x) > 0$ for any $x \in \Sigma$.
- Likelihood ratio $\ell_\delta = f_\delta^L / f_\delta^H$ is strictly decreasing, and satisfies $\max_{x \in \Sigma} |\ell_\delta(x) - 1| < \delta$
- Define the following likelihoods

$$\lambda_\delta(x) \equiv \frac{\Pr(x_{\delta,i} < x \mid L)}{\Pr(x_{\delta,i} < x \mid H)} = \frac{F_\delta^L(x)}{F_\delta^H(x)}$$

$$\Lambda_\delta(x) \equiv \frac{\Pr(x_{\delta,i} \geq x \mid L)}{\Pr(x_{\delta,i} \geq x \mid H)} = \frac{1 - F_\delta^L(x)}{1 - F_\delta^H(x)}$$

- $\lambda_\delta(x) > \ell_\delta(x) > \Lambda_\delta(x) > 0; \quad \lambda'_\delta(x) < 0, \Lambda'_\delta(x) < 0$

Market microstructure

- An asset that is worth 1 in H and 0 in L
- n informed traders decide to buy ($d_{n,i} = 1$) or not ($d_{n,i} = 0$).
- Each informed trader submits demand function $d_{n,i}(p)$.
- Trading volume is denoted by $m_n = \sum_{i=1}^{n} d_{n,i}$.
- Aggregate demand function $D(p) = \sum_{i=1}^{n} d_{n,i}(p)/n$
- Uninformed traders submit supply function $S(p)$
- $S(0.5) = 0$, $S' > 0$, $S(\bar{\Sigma}) = \bar{p} < 1$
- Auctioneer clears the market $D(p_n^*) = S(p_n^*)$
- $\mathrm{supp} P_n^* = [0.5, \bar{p}]$

Rational Expectations Equilibrium

For each realization of information profile $(x_{\delta,i})_{i=1}^{n}$, a rational expectations equilibrium consists of price p_n^*, trading volume m_n^*, demand functions $d_{n,i}(p)$, and posterior belief $r_{n,i}$ such that

- for any p, $d_{n,i}(p)$ maximizes i's expected payoff evaluated at $r_{n,i} = r(p_n, x_{\delta,i})$
- $r_{n,i}$ is consistent with p_n and $x_{\delta,i}$ for any i
- the auctioneer delivers the orders $d_{n,i}^* = d_{n,i}(p_n^*)$ and clears the market, $S(p_n^*) = m_n^*/n$, where $m_n^* = \sum_{i=1}^{n} d_{n,i}^*$

Informed trader's optimal behavior

- Trader i maximizes expected payoff: $r_{n,i} - p_n$ if buying and 0 otherwise.
- $p_n(m)$ denotes the price level such that $S(p_n) = m/n$
- i's optimal threshold policy:

$$d_{n,i}(p_n(m)) = \begin{cases} 1 & \text{if } x_{\delta,i} \geq \bar{x}(m) \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

where $\bar{x}$ the threshold level of private signal at which i is indifferent between buying and not.

Threshold rule and revealed information

Given the threshold rule, the information revealed by "buy" and "not-buy" actions are $\lambda_\delta(\bar{x})$ and $\Lambda_\delta(\bar{x})$.

When $p_n(m)$ realizes, the information revealed to a buying trader is:

$$\lambda_\delta(\bar{x}(m))^{n-m}\Lambda_\delta(\bar{x}(m))^{m-1} \tag{2}$$

The threshold is determined by:

$$\frac{1}{p_n(m)} - 1 = \lambda_\delta(\bar{x}(m))^{n-m}\Lambda_\delta(\bar{x}(m))^{m-1}\ell_\delta(\bar{x}(m)) \tag{3}$$

Upward sloping aggregate demand function

- Lemma 1: For sufficiently large n, $\bar{x}(m)$ is strictly decreasing in m and $D(p_n(m))$ is non-decreasing in m.
- Proof:

$$\frac{d\bar{x}_n}{dm} = \left.\frac{-\log\left(\Lambda_\delta(x)/\lambda_\delta(x)\right) - \{S'(p_n(m))p_n(m)(1-p_n(m))n\}^{-1}}{(n-m)\lambda'_\delta(x)/\lambda_\delta(x) + (m-1)\Lambda'_\delta(x)/\Lambda_\delta(x) + \ell'_\delta(x)/\ell_\delta(x)}\right|_{x=}$$

  Use $\lambda'_\delta < 0$, $\Lambda'_\delta < 0$, $\ell'_\delta < 0$, and $\lambda_\delta(x) > \Lambda_\delta(x)$.
- A higher price indicates that there are more traders who receive high signals $\rightarrow$ strategic complementarity

Existence of equilibrium

- Proposition 1:
  For sufficiently large n, there exists an equilibrium outcome (p_n^*, m_n^*) for each realization of $(x_{\delta,i})_{i=1}^n$.
- Proof:
  - Construct a reaction function $m' = \Gamma(m) \equiv D(p_n(m))n$: the number of traders with $x_{\delta,i} \geq \bar{x}(m)$.
  - Γ is non-decreasing, and thus Tarski's fixed point theorem applies.
- Multiple equilibria may exist. We focus on the minimum equilibrium outcome m_n^*.

Minimum outcome m_n^* as a first passage time

- A counting process $Y_o(x) \equiv \sum_{i=1}^{n} I_{X_{\delta,i} \geq x}$, where $X_{\delta,i}$ follows density f_δ^H
- M_n^* is equivalent to the first passage time m such that $Y_o(\bar{x}_n(m)) = m$.
- Change of variable $t = \bar{x}_n^{-1}(x) - 1$. ($t$ corresponds to $m - 1$ for $t = 0, 1, \ldots, n-1$.) Then, t follows $\tilde{f}_{\delta,n}(t) \equiv f_\delta^H(\bar{x}_n(t+1))|\bar{x}_n'(t+1)|$.
- Transform $Y_o(x)$ to $Y(t)$, satisfying $Y_o(x = \bar{x}_n(m)) = Y(t = m-1)$.
- M_n^* is the first passage time for $Y(t) = t$.

$Y(t)$ follows a Poisson process asymptotically as $n \to \infty$

- The number of traders who switch to buy during $(t, t+dt)$ follows a binomial distribution with population $n - Y(t)$ and probability $q_{\delta,n}(t)dt \equiv \tilde{f}_{\delta,n}(t)dt / F_{\delta}^{H}(\bar{x}_n(t+1))$
- $Y(0)$ follows a binomial distribution with population n and probability $q^{o}_{\delta,n} \equiv 1 - F_{\delta}^{H}(\bar{x}_n(1))$.
- Lemma 2: As $n \to \infty$, $Y(t)$ asymptotically follows a Poisson process with intensity:

$$\lim_{n \to \infty} \frac{\log \ell_{\delta}(\bar{x}_n(t+1))}{\ell_{\delta}(\bar{x}_n(t+1)) - 1}$$

Change-of-time for the first passage time distribution

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- We obtain $\mathbb{E}[Z(\tilde{\tau}_\psi)] = 1$ and $\mathbb{E}[\exp(\beta\tau_\psi)] = \{1 - \zeta(\beta,\psi)\}^c$, which is continuous w.r.t. ψ.

Explicit distribution of τ_1 conditional on $Y(0)$

Proposition 2: As $n \to \infty$ and $\delta \to 0$, M_n^* conditional on $Y(0) = c$ asymptotically follows

$$\Pr(M_n^* = m \mid Y(0) = c) = (c/m)e^{-m}m^{m-c}/(m-c)!, \quad m = c, c+1, \ldots$$

Moreover, the tail of the asymptotic distribution follows a power law with exponent 0.5, i.e., $\Pr(M_n^* > m) \propto m^{-0.5}$ for sufficiently large values of m.

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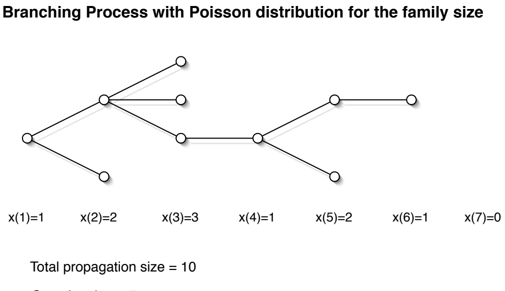
Proof: The stopping time of the Poisson process with intensity 1 is equivalent to the sum of a branching process with Poisson distribution with mean 1.



Unconditional distribution of τ_1

Proposition 3:
Suppose that $n(1 - F_\delta^H(\bar{x}_n(1)))$ converges to a positive constant ϕ_δ^o as $n \to \infty$. Then for sufficiently small δ and large n, the distribution function of M_n^* is arbitrarily close to:

$$\Pr(M_n^* = m) = \frac{\phi_\delta^o e^{-m-\phi_\delta^o}}{m!} (m + \phi_\delta^o)^{m-1}, \quad m = 0, 1, \ldots.$$

Moreover, M_n^* has a power-law tail distribution with exponent 0.5.

Intuition: Keynes' beauty contest

- "Critical" strategic complementarity
- The mean number of traders induced to buy by a buying trader is 1.
- Power law: M_n^* can occur at any order of magnitude
- Analogous to indeterminacy in the beauty contest
- At $n = \infty$,

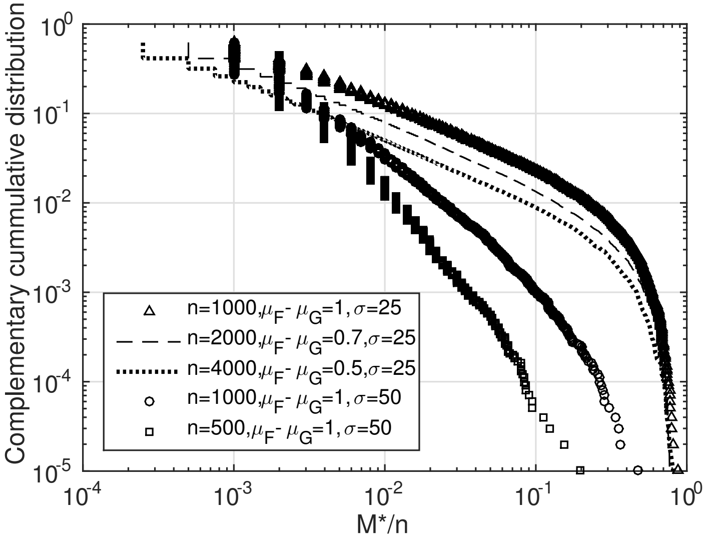
$$(1-\mu)\log\lambda_n(\bar{x}) + \mu\log\Lambda_n(\bar{x}) = 0$$

  for any $\mu = m/n$
- Power law: the distribution is scale-free

Numerical simulation: Specifications

- Price impact function $S(p) = 0.5 + 0.5(m/n)^{\gamma}$
- $\gamma = 0.5$: the square-root specification (Hasbrouck and Seppi 2001; Lillo, Farmer, and Mantegna 2003)
- X_i are drawn from a normal distribution $N(\mu, \sigma^2)$
- $\mu_H = 1$, $\mu_L = 0$, $\sigma = 25, 50$
- $N = 500, 1000$
- True state alternates between H and L
- Monte Carlo simulation with 100,000 draws

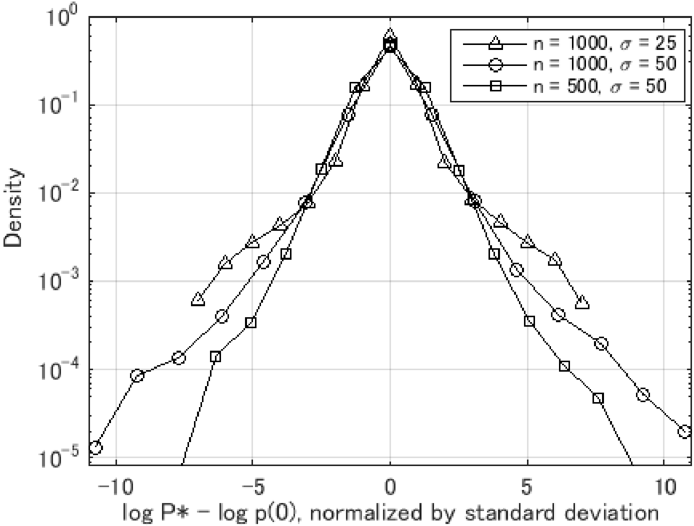
Simulated distribution of trading volume



Complementary cumulative distributions of M^*

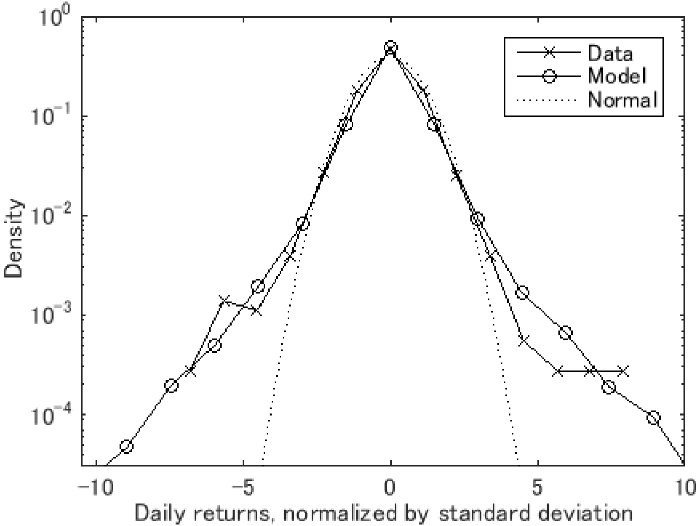
Thinner tails for some parameters: "sweeping of instability"

Simulated distributions of log P^*



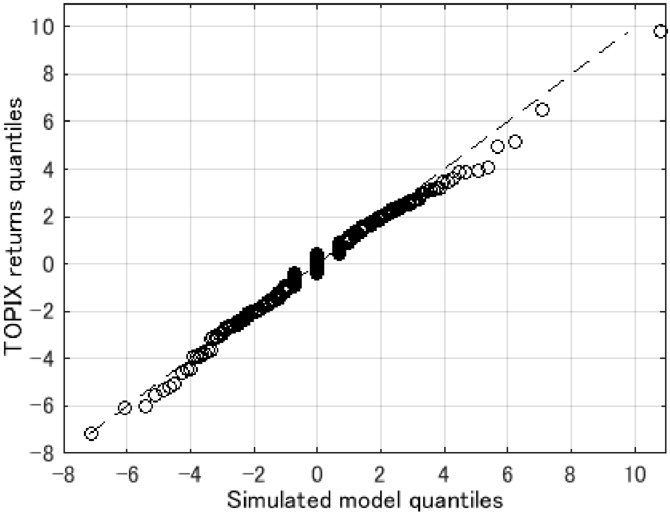
Semi-log density of returns $\log P^* - \log p(0)$

Stock Return Distribution: Model and data



Distributions of TOPIX daily returns, simulated returns $\log P^* - \log p(0)$, and a standard normal distribution

Stock Return Distribution: Q-Q plot



Quantile-to-quantile comparison of TOPIX daily returns and simulated returns

Discussion

- Informativeness of private signal is minimal ($\delta \to 0$)(e.g., unit time is infinitesimal)
- Traders are symmetric (unlike the herd behavior model)
- Information weight: The revealed likelihood of traders' actions may be discounted heterogeneously across traders
  - Classical herd behavior model is the case where trader i puts weight 1 for traders $1, \dots, i-1$ and 0 for traders $i+1, i+2, \dots$
  - Models based on traders' network provide a mechanism to generate such heterogeneous information weights
- Discreteness of actions is important for the private signal to be "hoarded"

Conclusion

- Criticality of trading-volume fluctuations emerges from the information aggregation among traders
- The power-law exponent for the volume is explained without parametric assumptions on environments
- Stock returns may inherit the non-Gaussian distribution of the volume

Digression: Power Laws

- Power exponent α (or Pareto exponent)
- Pareto distribution (1896), income and wealth $\alpha = 1.5$
- Zipf's law (1949), city size $\alpha = 1$
- Lotka (1926), "Law of scientific productivity", the number of papers authored by scientists

Empirical Power Laws

Mark E.J. Newman, "Power laws, Pareto distributions and Zipf's law", *Contemporary Physics*, Vol. 46, No. 5, September-October 2005, 323-351

1. frequency of use of words, 2.20
2. number of citations to papers, 3.04
3. number of hits on web sites, 2.40
4. copies of books sold in the US, 3.51
5. telephone calls received, 2.22
6. magnitude of earthquakes, 3.04
7. diameter of moon craters, 3.14
8. intensity of solar flares, 1.83
9. intensity of wars, 1.80
10. net worth of Americans, 2.09
11. frequency of family names, 1.94
12. population of US cities, 2.30

Models for generating power-law distributions (cf Newman)

Model 1: Inverses of stuff
Any quantity $x = y^{-\gamma}$, where y is a random variable that takes values around 0, has a power-law tail $p(x) \sim x^{-\alpha}$ where $\alpha = 1 + 1/\gamma$

Model 2: Generalized Central Limit Theorem

A normalized sum of independent random variables converges to a Lévy stable distribution with a tail parameter $\alpha \in (0, 2]$ (and three other parameters)

- Gaussian distribution is a special case with $\alpha = 2$. It is the only stable distribution with finite variance.
- Gaussian distribution is an attractor of distribution functions with finite variance (i.e., Central Limit Theorem)
- Lévy distribution with $\alpha < 2$ is an attractor of distribution functions with a power-law tail with exponent α
- Normalization: $N^{1/\alpha}$
- E(Sum/Maximum) converges to $1/(1\text{-}\alpha)$ for positive-valued distributions in a basin of attraction of a stable law $\alpha < 1$ (cf. Feller)

Cont'd; Stable laws

- First passage time in Brownian motion, $\alpha = 0.5$
  - Dimension analysis: independent increments + density only depending on x^2/t
- Holtsmark distribution (1919) of the gravitation force, $\alpha = 1.5$
  - Dimension analysis: density of mass relating to an inverse of cubed distance, gravity relating to an inverse of squared distance

Extreme Value Theory

The sample maxima $M_n = max(X_1, X_2, \ldots, X_n)$, properly normalized and centered, asymptotically follows the Generalized Extreme Value Distribution that nests:

- Weibull distribution
  - The maximum domain of attraction includes Uniform, Beta, ...
- Gumbel distribution
  - MDA: Exponential, Gamma, Normal, Lognormal, ...
- Fréchet distribution
  - MDA: Cauchy, Pareto, Loggamma, ...
  - has a power-law tail $x^{-\alpha}$

Model 3: Combinations of exponentials

- Combinations of exponentials; "logarithmic Boltzmann law"
  - If y is exponentially distributed $p(y) \sim e^{ay}$, then $x \sim e^{by}$ follows a power law $p(x) \sim x^{-1+a/b}$ (cf Newman)
    - If y is normally distributed, x follows a log normal.
  - (Maxwell-) Boltzmann distribution: velocities of particles of a gas follows an exponential
    - Kubo: Distribute money (energy) M to N persons (particles). # of possible sequences of numbered money and separators for persons: $(M+N-1)!$. # of possible ways to number money and separators: $M!$ and $N-1!$. Thus, # of configurations of the distribution is $W(N,M) \equiv (M+N-1)!/M!/(N-1)!$. Under the equal a priori probability postulate (fundamental postulate), the money distribution is $p(x) = W(N-1,x)/W(N,M) \sim (N/(M+N))(M/(M+N))^x$
    - Laplace's principle of indifference; Jaynes' principle of maximum entropy

Cont'd

- Multiplicative process with modifications
  - Reflective lower bound; Laplace's law on barometric density distribution
    - Random walk with negative drift and reflective lower bound has a stationary exponential distribution (Mandelbrot 1960; Gabaix 1999; Harrison "Brownian motion and stochastic flow systems" p.14)
  - Kesten process
  - Diffusion with killing (cf Oksendal)
- Yule process (rich-get-richer mechanism)
  - generates Yule distribution $p_x \propto Beta(x, \alpha) \sim x^{-\alpha}$
  - Ijiri and Simon, birth-and-death process, city size
  - Preferential attachment

Model 4: Critical Phenomena

- Phase transition and criticality
- Ising model for ferromagnet (vertex-model)
- Percolation of porous rocks (edge-model)
- Contact process
- Random-cluster models
- Erdos-Renyi random graph
- Renormalization
- Self-organized criticality, sand-pile model, Bak, Chen, Scheinkman, and Woodford (1994), percolation on Bethe lattice

Cont'd

- Fractals; self-similarity
- Scale-free; Macro-micro link
- Highly optimized tolerance (HOT), Fragmentation, etc

Theories for financial fat tails

- Statistical models (Subordinated process, some ARCH, Langevin equation, truncated Levy, etc)
- Agent-based (micro-founded) models
  - Herd behavior models (Scharfstein and Stein 1990; Banerjee 1992; Bikhchandani, Hirshleifer, and Welch 1992)
    - It explains herdings, but not fat-tails
  - Critical phenomena in statistical physics, network models, agent-based simulations (Bak, Paczuski, Shubik 1997; LeBaron, Arthur, and Palmer 1999; Lux and Marchesi 1999; Stauffer and Sornette 1999; Cont and Bouchaud 2000)
- **This paper shows a critical phenomenon in a herd behavior model**

A herd behavior model (Banerjee 1992)

- Two restaurants: A and B. 100 customers in line. Each customer observes the choices of customers before him
- Customers' prior belief is slightly in favor of A to B
- In reality, B is better than A
- Each customer draws a private information about the quality. 99 customers draw bad news about A
- The only customer who gets good news about A happens to be at the first in the line. He chooses A
- Second customer, observing the first customer's choice, chooses A regardless of his own information, because even though he draws a bad news about A, it cancels out with the first customer's **revealed information**
- All customers end up in the "wrong" restaurant A

Some modeling issues

- Herd in sequential move
  - Herding (everyone takes the same action)
  - Information cascade (agent's action is independent of its private information)
  - Choice set is "coarser" than information set
- Rational expectations equilibrium in a simultaneous-move game
  - Agreeing to disagree (Aumann 1976; Minehart and Scotchmer, GEB 1999)
  - Implementability (cf. Vives, Princeton UP 2008)
- Price impact function
  - No trade theorem (Milgrom and Stokey 1986)
  - Market microstructure (Kyle 1985; Avery and Zemsky, AER 1998; Gabaix et al, QJE 2006)

Related topics

- ϕ: degree of strategic complementarity
  - $\phi = 1$: "perfect" complementarity
  - Keynes' beauty contest: a trader's belief is affected proportionally by the average belief revealed
- Dynamical systems under $\phi = 1$ and discrete actions
  - "Neutral" dynamics; not strongly nonlinear
  - Weakly connected neural network (discrete action as a limit of logistic function); Globally coupled maps (GCM's)
- Role of "perfect" complementarity in macroeconomy
  - Monopolistic supply under duplicable and indivisible technology (CRS globally, IRS locally)
    - "Fragile" equilibrium
  - Monopolistic pricing under monetary neutrality
  - Balance-sheet contagion