Zipf's Law, Pareto's Law, and the Evolution of Top Incomes in the U.S.

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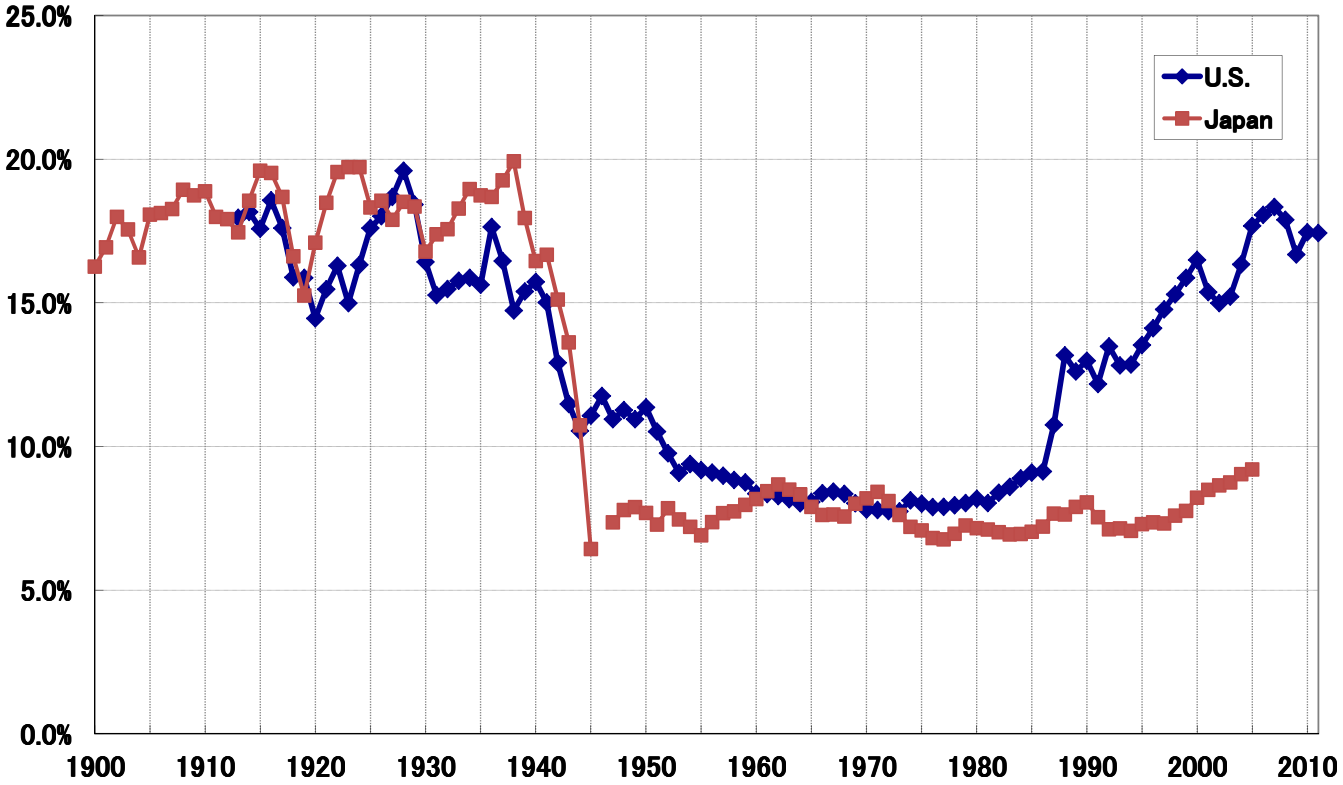
“We are the 99%”

Top 1% share

- "We are the 99%" in the "Occupy Wall Street" movement comes from the fact measured in Piketty and Saez (2003)
- Top 1% share measures how much percent of GNP goes to the richest 1% of the total population.
- Top 1% share in the U.S. has increased over three decades and now at the prewar level.



Note: Top 1% shares (Alvaredo et al., 2013).

Income distribution of the richest persons

- It is known that the tail part of income follows a Pareto distribution very well (Pareto's law of incomes)

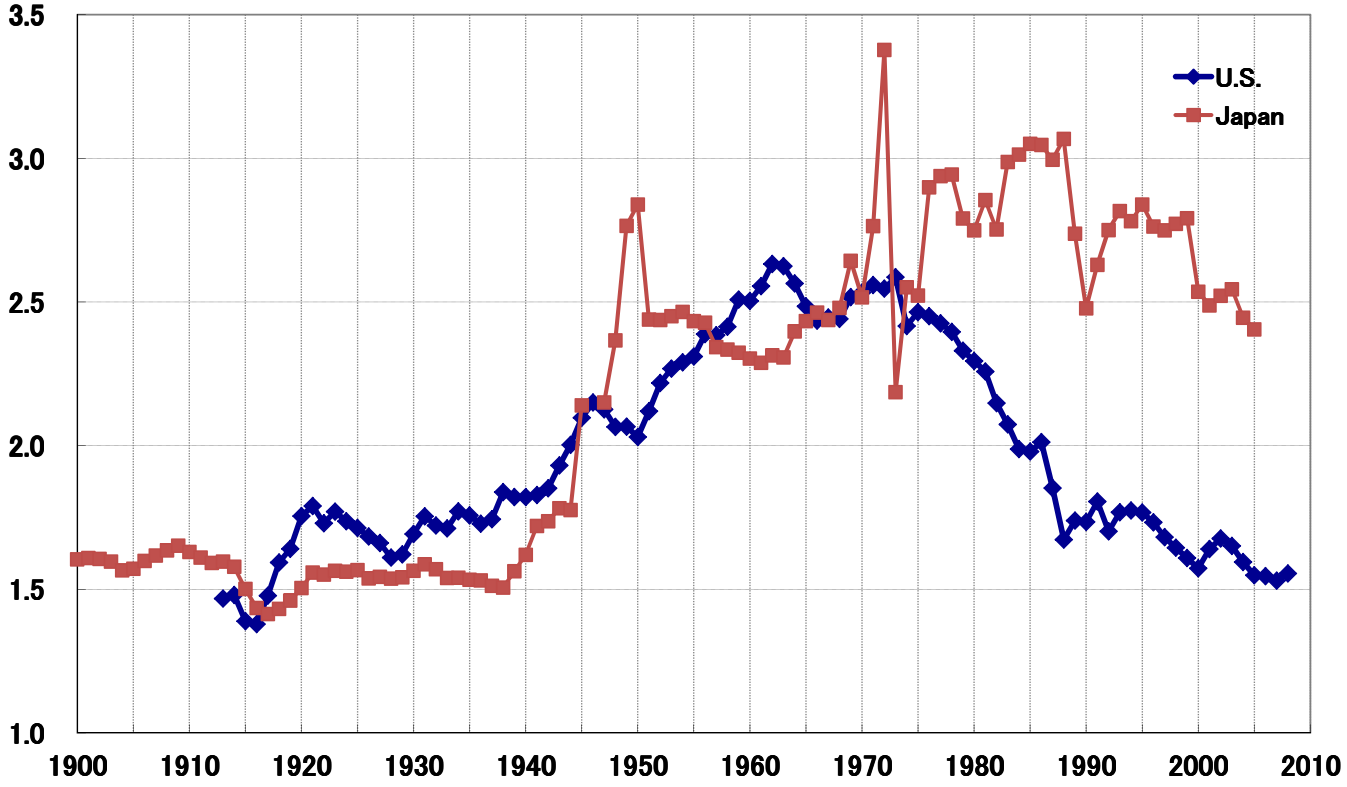
$$\Pr(x_i > x) \propto x^{-\lambda}, \quad \text{or,} \ \ln \Pr(x_i > x) = \mathrm{const} - \lambda \ln x.$$

  $\Pr(x_i > x)$: fraction of population whose income is more than x.
  λ: Pareto exponent.

- The Pareto distribution has a very thick tail.
  In the Pareto world, very rich persons such as Bill Gates possess a large portion of the national income than predicted under e.g., the lognormal distribution.

Pareto exponent of income distribution

- The Pareto exponent is negatively correlated with top 1% share. Especially, with the increase in top 1% share, the Pareto exponent has decreased.
  This implies that the dispersion of income within the top income group has widened.
- To understand the increase in the top 1% share, we need to understand why the Pareto exponent has decreased.



Note: Pareto exponents of income distributions (Alvaredo et al., 2013).

Purpose of the paper

- Much has been debated about the causes.
- We pay special attention to the decrease in the marginal income tax rate.
- The purpose:
  - develop a tractable dynamic general equilibrium model of income distribution,
  - and using the model, analyze how the decrease in the marginal income tax rate affects the concentration of income.

Firm side stylized facts 1.

We require our model to be consistent with the firm side stylized facts.

- Increase in top 1% share has been driven by the increase in CEO (or more broadly executive) pay.
- CEO's pay and asset strongly depend on the performance of his firm.
- In the standard neoclassical models, firm's performance is determined by the firm's productivity.

Therefore, the model of the concentration of income should be consistent with the stylized facts on firm's productivity.

Firm side stylized facts 2.

Zipf's law and Gibrat's law: stylized facts on firm size, which, in the standard models, are generated from firm's productivity shocks.

- Zipf's law: the firm size distribution follows a special case of Pareto distribution with exponent $\lambda = 1$.
- Gibrat's law: the growth of a firm size is a random walk.

We construct our model consistent with these laws.

Contribution of the paper:

- present a parsimonious neoclassical growth model that generates Zipf's law and Gibrat's law of firms and Pareto's law of incomes from idiosyncratic firm-level productivity shocks.
- and using the model, analyze how the decrease in the marginal income tax rate affects the concentration of income.

Related literature

Several papers have built models of the Pareto income distribution. Two approaches:

1. Models that generate Pareto's law of incomes from other distributions: Gabaix and Landier (2008) and Jones and Kim (2012)
   - They assume that the firm size distribution follows Zipf's law and that the CEO's talent follows a certain distribution.
   - They show that the CEO pay follows a Pareto distribution.

   They need to assume certain types of distributions.
2. Models that generate Pareto's law from idiosyncratic shocks:
   - Household models: Nirei and Souma (2007) and Benhabib et al. (2011). Shocks on household's asset return.
   - DGE models: Nirei (2009), Toda (2012), Kim (2013). Shocks on productivity or shocks on capital.

   They does not relate the Pareto's law with Zipf's law.

We adopt the second approach and generate the Pareto's law and Zipf's law from firm-level productivity shocks.

Outline

Basic mechanism 1.

Here, we explain a basic mechanism which generates a Pareto distribution.
We show

- a directed random walk with a reflecting wall generates an exponential distribution as a stationary distribution.
- by changing variables, an exponential distribution can be transformed to a Pareto distribution.

Suppose a one-dimensional directed random variable x_t^i:

- The initial point is $x_0^i = 0$.
- $x_{t+1}^i = x_t^i + \epsilon_t^i$, where

$$\epsilon_t^i = \begin{cases} +1 & \text{with prob. } p < 0.5 \\ -1 & \text{with prob. } 1 - p > 0.5 \end{cases}$$

It is "directed" in the sense that x_t^i is pulled to the minus direction.

In this case, because x_t^i diffuses over time, the stationary distribution of x_t^i does not exist.

Basic mechanism 2.

We add a reflecting wall to the directed random walk:

- $\epsilon_t^i = +1$ if $x_t^i = 0$.

- In this case, the stationary distribution of $x_{t+1}^i = x_t^i + \epsilon_t^i$ exists and is an exponential distribution (more precisely, geometric distribution):

$$\Pr(x_t^i > x) = \exp\left\{-\frac{x - x_0}{\beta}\right\}$$

- Next, suppose y_t^i that satisfies $\ln y_{t+1}^i = \ln y_t^i + \epsilon_t^i$.
  The stationary distribution of y_t^i is a Pareto distribution:

$$\Pr(y_t^i > y) = \left(\frac{y}{y_0}\right)^{-\beta}$$

We embed this mechanism into a DGE model and derive Pareto's law of incomes and Zipf's law of firms.

Outline

Environment

Continuous-time incomplete market model.
Households:

- Perpetual youth model with constant population L. ν: prob. to die.
- Entrepreneurs N and workers $L - N$
  - Each provides one unit of labor.
  - Only entrepreneurs can manage firms.
  - Entrepreneurs become workers with prob. p_f (a setup needed for the quantitative analysis).
  - No borrowing constraint.
- They choose the consumption and asset portfolio optimally to maximize

$$U = \int_0^\infty \ln c_t^i e^{-(\beta+\nu)t} dt \quad (\beta\text{: discount rate, and } \nu\text{: prob. to die}).$$

Portfolio choice problem of households

- A Worker can hold
  - human assets that consist of wage incomes
  - and risk-free market portfolio,
    which consist of the market portfolio of firms' stocks.
- An Entrepreneur can hold
  - human assets,
  - risk-free financial assets,
  - and risky stocks of his firm.
- In our model, the riskiness and return of entrepreneur's risky stocks are ex-ante identical across entrepreneurs.
- Taxes are imposed on the dividends of the risk-free market portfolio and risky stocks. The tax rates are different between these assets.
- When a household holds the risk-free market portfolio, he has to pay transaction cost that is proportional to the amount of the market portfolio.
  Due to the setup, the expected return of the risky assets exceeds that of the risk-free market portfolio.

Asset process of a household and Pareto's law of incomes

- Under the log utility, a household consume a constant fraction of the total asset, and an entrepreneur hold a constant fraction of the total asset for risky assets.
- As a result, detrended total assets of an entrepreneur and a worker ($\tilde{a}_{e,t}$ and $\tilde{a}_{w,t}$.) evolve as follows:

$$d \ln \tilde{a}_{e,t} = \left(\mu_{ae,t} - g - \frac{\sigma_{ae,t}^2}{2} \right) dt + \sigma_{ae,t} dB_{i,t}, \quad d \ln \tilde{a}_{w,t} = (\mu_{aw} - g)\, dt.$$

  g: the trend growth rate of the economy, $\sigma_{ae,t} dB_{i,t}$: volatility from holding risky assets.
- $\ln \tilde{a}_{i,t}$ is a random walk process.
  By the perpetual youth setup, a constant fraction of households is replaced by new born households, whose assets are reset to the relatively low initial level.
  $\Longrightarrow$ Due to a similar (but different) mechanism as above, the stationary distribution of total assets follows a Pareto distribution at the upper tail.
- Because income is proportional to total asset, the stationary distribution of incomes also follows a Pareto distribution at the upper tail.

Pareto distribution of firms

- Heterogeneous: one firm for one entrepreneur.
- The Dixit-Stiglitz type monopolistic competition: the heterogeneous firms produce differentiated goods.
- Each firm issues stocks and owns the physical capital as in McGrattan and Prescott (2005).
- Production function: $y_{j,t} = z_{j,t} k_{j,t}^{\alpha} \ell_{j,t}^{1-\alpha}$.
- Detrended productivity of a firm evolves as follows:

$$d \ln \tilde{z}_{j,t} = \left(\mu_z - g_z - \frac{\sigma_z^2}{2} \right) dt + \sigma_z dB_{j,t}.$$

- $d \ln \tilde{z}_{j,t}$ is a random walk process, i.e., Gibrat's law holds.
- We assume that there exists a lower bound to the employment level of a firm.
  $\Longrightarrow$ Due to the same mechanism as explained above, the stationary distribution of the detrended productivity follows a Pareto distribution.
- Under a standard setup, the capital and labor also follow Pareto distributions.
- The lower is the lower bound, the closer is the Pareto exponent of the firm size distribution to 1, i.e., Zipf's law holds.

Structure of the model

- Heterogeneous households consume and invest a constant fraction of the total asset.
- Heterogeneous firms choose capital and employment proportional to the productivity.
- Government: collected taxes are allocated to households in a lump sum way (the government expenditure is set to zero).

$\Longrightarrow$ Dynamics of the aggregate economy can be solved separately from the heterogeneities of households and firms.

- The aggregate dynamics is the basically the same as the standard neoclassical model. $\Longrightarrow$ Tractable.
- In the quantitative analysis, we solve the aggregate dynamics using the shooting algorithm.

Outline

Our interpretation of the increase in top 1% share

Using the model, we analyze how the changes in the top marginal tax rates affect top 1% share.

We assume

- the changes in the top marginal tax rates occurred suddenly at 1975.
- the changes are permanent.

Our interpretation of the increase in top 1% share:

- the tax on risky assets is equal to the top marginal ordinary income tax that is imposed on the CEO pay.
- the tax on risk-free assets is equal to the sum of corporate and capital gains taxes.
- In the data, the magnitude of the reduction in the top marginal ordinary tax is larger than that of corporate and capital gains taxes.
- $\Longrightarrow$ after the tax changes, the expected return of risky stocks becomes relatively high.
- $\Longrightarrow$ Entrepreneurs increase the holdings of risky stocks.
- $\Longrightarrow$ Volatility of entrepreneur's asset becomes higher.
- $\Longrightarrow$ The Pareto exponent of income distribution decreases, and top 1% share increases.

Tax rates

	pre-1975	post-1975
ordinary income tax, τ^{ord}	0.75	0.40
corporate income tax, τ^{corp}	0.50	0.35
capital gain tax, τ^{cap}	0.25	0.25
τ^{e}	0.75	0.40
τ^{f}	0.63	0.51

- The figures in the first three rows are taken from Saez et al. (2012).
  - Tax on risky assets τ^{e} is the same as the ordinary income tax τ^{ord}.
  - Tax on risk-free market portfolio τ^{f} is calculated from

$$1 - \tau^{f} = (1 - \tau^{\mathrm{cap}})(1 - \tau^{\mathrm{corp}}).$$

Calibration 1.

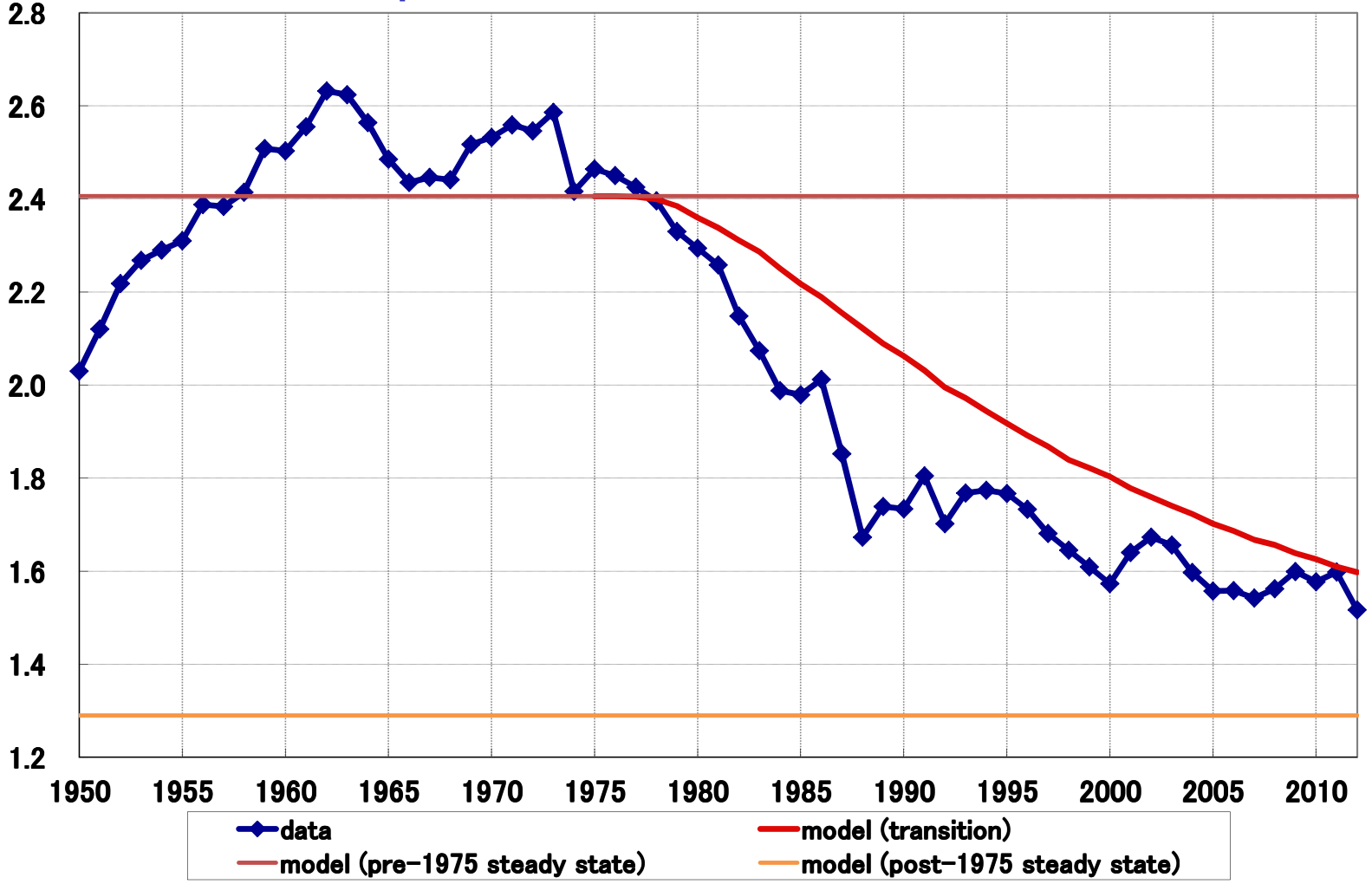
β	discount rate	0.04
ν	prob. of death	1/50
α	capital share	1/3
δ	depreciation rate	0.1
g	steady-state growth rate	0.02
ρ	elasticity of substitution	0.7
p_f	prob. of entrepreneur's quit	1/20
ℓ_{min}	min. level of employment	1
L	fraction of population	1.0
N	fraction of entrepreneurs	0.05

- The first five rows use standard values.
- $\rho = 0.7$: a parameter in the CES function (then, profit rate is 30%). It is lower than the standard one.
  - This is because we interpret the incentive pay of CEOs in the real world as dividends from risky asset in the model.
- $\ell_{\text{min}} = 1$: the min. emp. level of a firm is 1 person.
- $L/N = 20$: average emp. level of a firm is 20 persons (consistent with Davis et. al., 2007).
- Under these parameters, the Pareto exponent of firm size dist. is $1/(1 - 0.05) \approx 1.0526$, which is close to Zipf's law.

Calibration 2.

- The volatility of productivity shocks σ_z is calibrated from the employment volatility of public firms (Davis et al., 2007).
- Transaction cost parameter ι is calibrated to match the Pareto exponent in the pre-1975 steady state with 2.4 that is close to the data around 1975 (for the reasonableness of ι, see Table 3 of the paper).

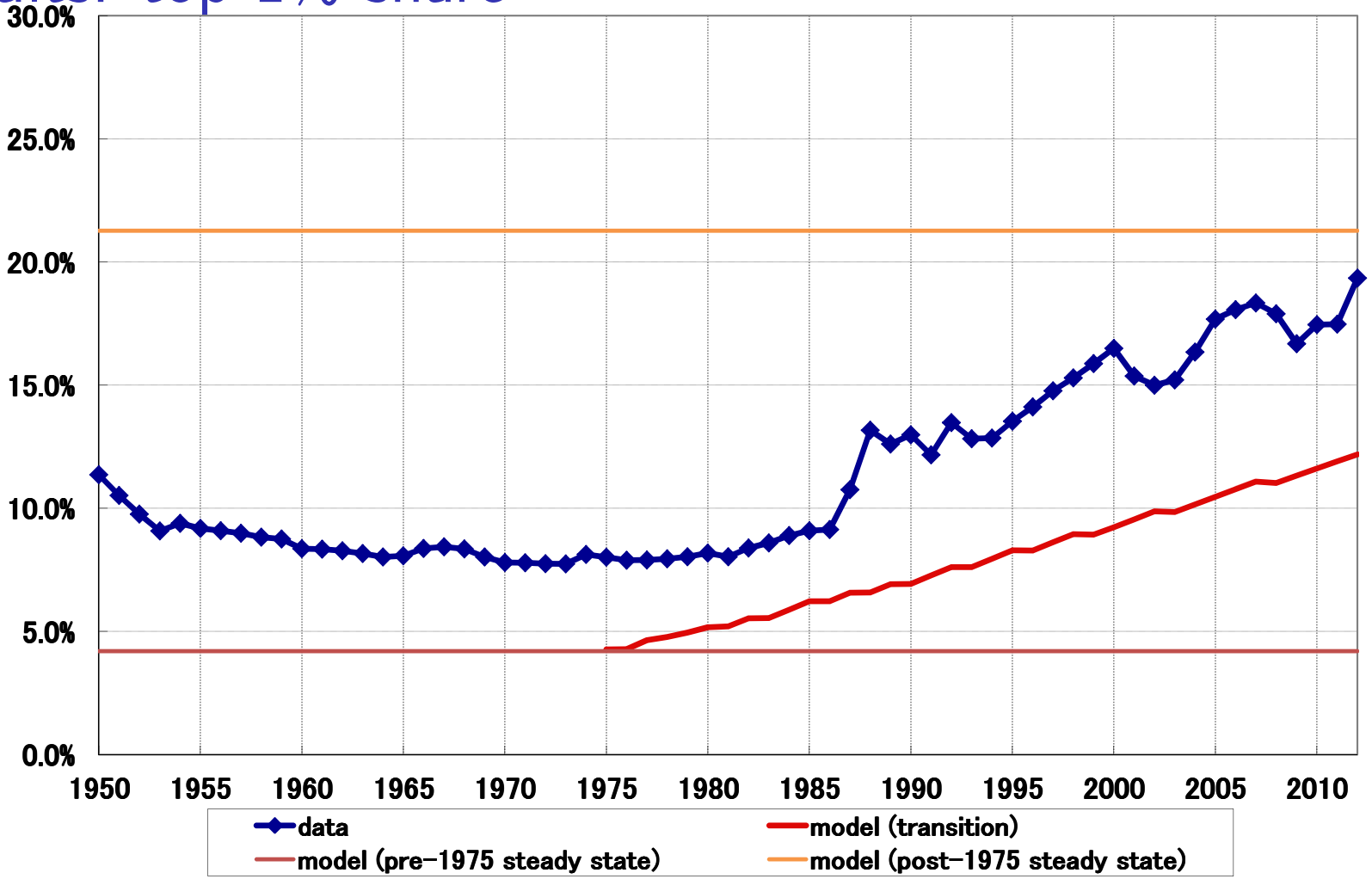
Results: Pareto exponent



Source: Data are taken from Alvaredo et al. (2013).

- Although it is obvious that the model matches the initial level of the Pareto exponent (we set ι to match), it is not obvious that the model also matches the changes.

Results: top 1% share



Source: Data are taken from Alvaredo et al. (2013).

- Our model can capture the trend in top 1% share. However, the level of model's top 1% share is lower.
- Perhaps, other factors such as the differences in talents account for the gap between them.

Outline

Conclusion

- We have proposed a model of asset and income inequalities that explains Zipf's law of firms and Pareto's law of incomes both from the idiosyncratic productivity shocks of firms.
- Using the model, we have analyzed to what extent the changes in tax rates account for the recent evolution of top incomes in the U.S.
  - The model matches with the decline in the Pareto exponent of income.
  - The model also matches with trend in top 1% share, while it does not match with the level of the top 1% share.